$$X_u = \frac{1}{(1+u^2+v^2)^2}\left[2(v^2-u^2+1), -4uv, 4u\right]$$

$$X_v = \frac{1}{(1+u^2+v^2)^2}\left[-4uv, 2(u^2-v^2+1), 4v\right]$$

$$\bar{E} = \frac{4(v^2-u^2+1)^2+16u^2v^2+16u^2}{(1+u^2+v^2)^4} = \frac{4(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = \frac{4}{(1+u^2+v^2)^2} = \langle X_u, X_u \rangle$$

$$\bar{F} = \langle X_u, X_v \rangle = 0$$

$$\bar{G} = \langle X_v, X_v \rangle = \frac{16u^2v^2+16v^2+4(u^2-v^2+1)^2}{(1+u^2+v^2)^4} = \frac{4}{(1+u^2+v^2)^2}$$

$$\therefore \lambda^2 = \frac{4}{(1+u^2+v^2)^2}$$

Recall: ① Every regular surface is locally conformal to the plane (flat plane) — 4/18

* isothermal, $E=G$, $F=0$. $I_S(\alpha') = \lambda^2 I_{IP}(\beta')$

② Any two regular surface are isothermal.

4-3 Gauss Theorem and equations of compatibility

S is a regular surface in $\mathbb{R}^3$, $\{X_u, X_v, N\}$ forms a basis for $\mathbb{R}^3$ at each point $p \in S$

$$⓪ \begin{cases} X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L_1 N \\ X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + L_2 N \\ X_{vu} = \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + L_3 N \\ X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + L_4 N \end{cases}$$

$\langle N, X_{uu} \rangle = L_1$ ($\because \langle N, X_u \rangle = 0, \langle N, X_v \rangle = 0, \langle N, N \rangle = 1$) $= e$

$\langle N, X_{uv} \rangle = L_2 = f = L_3 = \langle N, X_{vu} \rangle$

$\langle N, X_{vv} \rangle = L_4 = g$

$\langle X_u, X_{uu}\rangle \underset{(0)}{=} \Gamma^1_{11} E + \Gamma^2_{11} F$

$\langle X_u, X_{uv}\rangle \underset{(0)}{=} \Gamma^1_{12} E + \Gamma^2_{12} F$

$\langle X_u, X_{vu}\rangle \underset{(0)}{=} \Gamma^1_{21} E + \Gamma^2_{21} F$

$\Rightarrow \begin{cases} \Gamma^1_{12} = \Gamma^1_{21} \\ \Gamma^2_{12} = \Gamma^2_{21} \end{cases}$ Hence $\Gamma^k_{ij} = \Gamma^k_{ji}$

Γ^k_{ij} are called the christaffel symbols of X

<u>Fact</u>: Γ^k_{ij} can be computed by using the 1st fundemental Form alone

$$\Gamma^1_{11} = \frac{1}{2(EG-F^2)}(GE_u - 2FF_u + FE_v), \quad \text{Let } \Delta = EG-F^2$$

$$\Gamma^2_{11} = \frac{1}{2\Delta}[2EF_u - EE_v - FE_u]$$

$$\Gamma^1_{12} = \frac{1}{2\Delta}[GE_v - FG_u] = \Gamma^1_{21}$$

$$\Gamma^2_{12} = \frac{1}{2\Delta}[EG_u - FE_v] = \Gamma^2_{21}$$

$$\Gamma^1_{22} = \frac{1}{2\Delta}[2GF_v - GG_u - FG_v]$$

where $\frac{\partial E}{\partial u} = E_u = \langle X_u, X_u\rangle_u$

$$\Gamma^2_{22} = \frac{1}{2\Delta}[EG_v - 2FF_v + FG_u]$$

$\langle X_u, X_{uu}\rangle = \Gamma^1_{11} E + \Gamma^2_{11} F = \frac{1}{2}\langle X_u, X_u\rangle_u = \frac{1}{2} E_u$

$\langle X_v, X_{uu}\rangle = \Gamma^1_{11} F + \Gamma^2_{11} G = \underset{F_u}{\underset{\shortparallel}{\langle X_v, X_u\rangle_u}} - \underset{\frac{1}{2}E_v}{\underset{\shortparallel}{\langle X_u, X_{vu}\rangle}}$

$\therefore \begin{cases} \Gamma^1_{11} E + \Gamma^2_{11} F = \frac{1}{2} E_u \\ \Gamma^1_{11} F + \Gamma^2_{11} G = F_u - \frac{1}{2} E_v \end{cases}$

$\Rightarrow \Gamma^2_{11} = \frac{1}{GE-F^2}[EF_u - \frac{1}{2}EE_v - \frac{1}{2}FE_u]$ $\quad -(1)F + (2)E$

$\Gamma^1_{11} = \frac{1}{GE-F^2}[\frac{1}{2}GE_u + \frac{1}{2}FE_v - FF_u]$ $\quad (1)G - (2)F$

H.W

$\langle X_u, X_{uv}\rangle = \Gamma^1_{12} E + \Gamma^2_{12} F = \frac{1}{2}\langle X_u, X_u\rangle_v = \frac{1}{2} E_v$

$\langle X_v, X_{uv}\rangle = \Gamma^1_{12} F + \Gamma^2_{12} G = \frac{1}{2} G_u$

$\langle X_u, X_{vv}\rangle = \Gamma^1_{22} E + \Gamma^2_{22} F = F_v - \frac{1}{2} G_u$

$\langle X_v, X_{vv}\rangle = \Gamma^1_{22} F + \Gamma^2_{22} G = \frac{1}{2} G_v$

$\Rightarrow$

$\Gamma^1_{12} = \frac{1}{EG-F^2}\cdot\frac{1}{2}[GE_v - FG_u]$

$\Gamma^2_{12} = \frac{1}{EG-F^2}\cdot\frac{1}{2}[EG_u - FE_v]$

$\Gamma^1_{22} = \frac{1}{EG-F^2}\cdot\frac{1}{2}[2GF_v - GG_u - FG_v]$

$\Gamma^2_{22} = \frac{1}{EG-F^2}\cdot\frac{1}{2}[EG_v + FG_u - FF_v]$

Note: $\langle X_u, X_u\rangle = g_{11}$, $\langle X_u, X_v\rangle = g_{12}$, $\langle X_v, X_v\rangle = g_{22}$

$$\sum_k \Gamma_{ij}^k = \sum_k \frac{1}{2} g^{kl}\{\underbrace{g_{jl,i}}_{\frac{\partial}{\partial x_i}(g_{jl})} + g_{li,j} - g_{ij,l}\}, \quad g^{kl} = (g_{kl})^{-1}$$

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \to g \text{ is called Riemannian metric}$$

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{EG-F^2}\begin{bmatrix} G & F \\ -F & E \end{bmatrix}$$

$$\Gamma_{11}^k = \frac{1}{2}(g^{kl})\{g_{1l,1} + g_{l1,1} - g_{11,l}\}$$

$$k=1,\ \Gamma_{11}^1 = \sum_{l=1}^{2} \frac{1}{2}(g^{1l})\{2g_{1l,1} - g_{11,l}\}, \quad \because g_{11} = E,\ g_{11,1} = E_u$$

$$= \frac{1}{2}(g^{11})^{-1}\{2E_u - E_u\} + \frac{1}{2}(g^{12})^{-1}\{2F_u - F_v\}$$

$$(g^{11})^{-1} = \frac{G}{EG-F^2}, \quad (g^{12})^{-1} = \frac{-F}{GG-F^2}$$

From above computation, we observe that

① The christoffel symbols are in term of the coefficients of the 1^{st} fundamental form.

② All geometric concepts and properties expressed in term of christoffel symbols are invariant under isometric.

Codazzi — Mainardi equation

Let X be a parametrization of S with 1st f. form E, F, G and 2^{nd} f. form, e, f, g

$$\begin{cases} e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2 \\ f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{21}^1) - g\Gamma_{12}^2 \end{cases}$$

$$e_v = \langle N, X_{uu}\rangle_v = \frac{\partial}{\partial v}\langle N, X_{uu}\rangle$$

$$f_u = \langle N, X_{uv}\rangle_u = \frac{\partial}{\partial u}\langle N, X_{uv}\rangle, \quad \Gamma_{ij}^k = C(E, G, F)$$

HW: Find christoffel symbols Γ_{ij}^k for

$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$ $\quad u \in (0, 2\pi)$, $v \in \mathbb{R}$

$f > 0$, surface revolution

$\Gamma_{11}^1 = 0 \quad \Gamma_{11}^2 = \frac{-ff'}{f'^2 + g'^2}$

$\Gamma_{12}^1 = \frac{f'}{f} \quad \Gamma_{12}^2 = 0$

$\Gamma_{22}^1 = 0 \quad \Gamma_{22}^2 = \frac{f'f'' + g'g''}{f'^2 + g'^2}$

Let X be parametrization of S with 1st f. form E, F, G and 2nd f. form e, g, f

$$\begin{cases} e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} & \text{— (A1)} \\ f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} & \text{— (A2)} \end{cases}$$

where $e_v = \frac{\partial e}{\partial v}$, $f_u = \frac{\partial f}{\partial u}$, $g_u = \frac{\partial g}{\partial u}$, ...

$K \equiv$ Gauss Curvature

$$EK = (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})^2 \quad \text{— (B1)}$$

$$FK = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^1_{22}\Gamma^2_{11} \quad \text{— (B2)} \qquad (\Gamma^k_{ij})_u = \frac{\partial}{\partial u}(\Gamma^k_{ij})$$

$$FK = (\Gamma^2_{12})_v - (\Gamma^2_{22})_u - \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11} \quad \text{— (B3)} \qquad (\Gamma^k_{ij})_v = \frac{\partial}{\partial v}(\Gamma^k_{ij})$$

$$GK = (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{11}\Gamma^1_{22} \quad \text{— (B4)}$$

where (A1) and (A2) are called Codazzi-Mainardi equations, and (B1), (B2), (B3), (B4) called Gauss equations (formula)

Pf: Let $\{X_u, X_v, N\}$ form a basis.

$$\begin{cases} (X_{uu})_v - (X_{uv})_u = 0 & \text{— ①} \\ (X_{vv})_u - (X_{vu})_v = 0. & \text{— ②} \\ N_{uv} - N_{vu} = 0 & \text{— ③} \end{cases}$$

By ⓪, we have the following

$$\begin{cases} A_1 X_u + B_1 X_v + C_1 N = 0 & \text{— ①'} \\ A_2 X_u + B_2 X_v + C_2 N = 0 & \text{— ②'} \\ A_3 X_u + B_3 X_v + C_3 N = 0 & \text{— ③'} \end{cases}$$

where A_i, B_i, C_i, $i = 1, 2, 3$ are function of E, F, G, e, f, g and their deritives

$\Rightarrow A_i = B_i = C_i = 0$.

Look at $(X_{uu})_v - (X_{uv})_u = 0$.

$$= (\Gamma^1_{11})_v X_u + \Gamma^1_{11} X_{uv} + (\Gamma^2_{11})_v X_v + \Gamma^2_{11} X_{vv} + e_v N + e N_v$$
$$- [(\Gamma^1_{12})_u X_u + \Gamma^1_{12} X_{uu} + (\Gamma^2_{12})_u X_v + \Gamma^2_{12} X_{uv} + f_u N + f N_u]$$

$$= ((\Gamma^1_{11})_v X_u + \Gamma^1_{11}\Gamma^1_{12} X_u + \Gamma^1_{11}\Gamma^2_{12} X_v + (\Gamma^2_{11})_v X_v + \Gamma^2_{11}\Gamma^1_{22} X_u + \Gamma^2_{11}\Gamma^2_{22} X_v$$
$$+ \Gamma^1_{11} f N + \Gamma^2_{11} e N + e_v N + e(a_{12} X_u + a_{22} X_v)$$

$=(\Gamma_{11}^1)_v X_u + \Gamma_{11}^1(\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN) + (\Gamma_{12}^2)_v X_v + \Gamma_{11}^2(\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + gN)$
$+ e_v N + e(a_{12}X_u + a_{22}X_v) - [(\Gamma_{12}^1)_u X_u + \Gamma_{12}^1(\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + eN)$
$+ (\Gamma_{12}^2)_u X_v + \Gamma_{12}^2(\Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + fN) + f_u N + f(a_{11}X_u + a_{21}X_v) = 0$

Look at X_u: the coefficient of X_u in $(X_{uu})_v - (X_{uv})_u = 0$.

$A_1 = 0$.

$(\Gamma_{11}^1)_v + \Gamma_{11}^1\Gamma_{12}^1 + \Gamma_{11}^2\Gamma_{22}^1 + e a_{12} - [(\Gamma_{12}^1)_u + \Gamma_{12}^1\Gamma_{11}^1 + \Gamma_{12}^2\Gamma_{21}^1 - f a_{11}] = 0$

$e a_{12} - f a_{11} = (\Gamma_{12}^1)_u + \Gamma_{12}^2\Gamma_{21}^1 - (\Gamma_{11}^1)_v - \Gamma_{11}^2\Gamma_{22}^1$

recall: $a_{11} = \dfrac{fF - eG}{EG - F^2}$, $a_{12} = \dfrac{gF - fG}{EG - F^2}$,

$$e a_{12} - f a_{11} = \frac{egF - efG - f^2F + feG}{EG - F^2} = \frac{F(eg - f^2)}{EG - F^2} = FK \Rightarrow \text{we have } \bar{B_2}$$

$B_1 = 0$. (X_v)

$\Gamma_{11}^1\Gamma_{12}^2 + (\Gamma_{12}^2)_v + \Gamma_{11}^2\Gamma_{22}^2 + e a_{22} - \Gamma_{12}^1\Gamma_{11}^2 - (\Gamma_{12}^2)_u - f a_{21} - \Gamma_{12}^2\Gamma_{21}^2 = 0$

$$e a_{22} - f a_{21} = e\frac{fF - gE}{EG - F^2} - f\frac{eF - fE}{EG - F^2} = \frac{efF - egE - efF + f^2E}{EG - F^2} = \frac{E(f^2 - eg)}{EG - F^2}$$

$$= -EK = \Gamma_{12}^2\Gamma_{21}^2 + (\Gamma_{12}^2)_u + \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - (\Gamma_{12}^2)_v \quad \bar{B_1}$$

Look at N: the coefficient of N in $(X_{uu})_v - (X_{uv})_u = 0$, $C_1 = 0$.

$\Gamma_{11}^1 f + \Gamma_{11}^2 g + e_v - \Gamma_{12}^1 e - \Gamma_{12}^2 f - f_u = 0$

$\Rightarrow e_v - f_u = \Gamma_{12}^1 e + \Gamma_{12}^2 f - \Gamma_{11}^1 f - \Gamma_{11}^2 g \quad — \bar{A_1}$

Similarly, $(X_{vv})_u - (X_{vu})_v = 0$. $A_2 X_u + B_2 X_v + C_2 N = 0$

$A_2 = 0 \Rightarrow \bar{B_4}$

$B_2 = 0 \Rightarrow \bar{B_3}$

$C_2 = 0 \Rightarrow \bar{A_2}$

Theorem: (Gauss Egregium) $\longmapsto$ extremly important 2018.04.26

Gauss Remarkable Theorem

The Gauss Curvature is invariant under local isometry.

$(N_u)_v = (N_v)_u$

$A_3 X_u + B_3 X_v + C_3 N = 0$

$$\begin{cases} N_u = a_{11} X_u + a_{21} X_v \\ N_v = a_{12} X_u + a_{22} X_v \end{cases}$$

$(a_{11} X_u + a_{21} X_v)_v - (a_{12} X_u + a_{22} X_v)_u = 0$

$\underbrace{a_{11,v}}_{= \frac{\partial a_{11}}{\partial v}} X_u + a_{11} \underbrace{X_{uv}}_{\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + f N} + a_{21,v} X_v + a_{21} X_{vv} - [a_{12,u} X_u + a_{12} X_{uu} + a_{22,u} X_v + a_{22} X_{uv}] = 0$

$= (a_{11,v} + a_{11}\Gamma_{12}^1 + a_{21}\Gamma_{22}^1) X_u + (a_{21,v} + a_{11}\Gamma_{12}^2 + a_{21}\Gamma_{22}^2) X_v + (a_{11} f + a_{21} g) N$

$- [(a_{12,u} + a_{12}\Gamma_{11}^1 + a_{22}\Gamma_{12}^1) X_u + (a_{22,u} + a_{12}\Gamma_{11}^2 + a_{22}\Gamma_{12}^2) X_v + (a_{12} e + a_{22} f) N] = 0.$

$C_3 = 0$, "Look at N"

$a_{11} f + a_{21} g = a_{12} e + a_{22} f \Rightarrow a_{12} e + (a_{22} - a_{11}) f - a_{21} g = 0 \quad \left(\begin{matrix} E, F, G \\ e, f, g \end{matrix}\right)$

$$a_{11} = \frac{fE - eG}{EG - F^2} \qquad a_{21} = \frac{eF - fE}{EG - F^2}$$

$$a_{12} = \frac{gF - fG}{EG - F^2} \qquad a_{22} = \frac{fF - gE}{EG - F^2}$$

$A_3 = 0$, "X_u"

$a_{11,v} - a_{12,u} = a_{12}\Gamma_{11}^1 + a_{21}\Gamma_{22}^1 - a_{11}\Gamma_{12}^1 - a_{21}\Gamma_{22}^1$

L.H.S

$\Gamma_{ij}^k = C(E, F, G, E_u, E_v, F_u, F_v, G_u, G_v)$

plug into ◎ codazzi-equation

$$\left(\frac{fF - eG}{EG - F^2}\right)_v - \left(\frac{gF - fG}{EG - F^2}\right)_u$$

$$\Rightarrow \frac{(EG - F^2)(f_v F + f F_v - e_v G - e G_v) - (fF - eG)[E_v G + E G_v - 2F F_v]}{(EG - F^2)^2} - \frac{\cdots}{\cdots}$$

$\overline{A_1}$ $\overline{A_2}$

$B_3 = 0$, "X_v" $\leftrightarrow$ codazzi - Mainardi equation.

$(X_{uu})_v = (X_{uv})_u$ $\qquad$ $(X_{vv})_u = (X_{uv})_v$

$A_1 = 0$ $\qquad$ $A_2 = 0$

$B_1 = 0$ $\qquad$ $B_2 = 0$

$C_1 = 0$ $\qquad$ $C_2 = 0$.

$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2$ — ①

$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2$ — ②

where coordinate Nbd contains. no umbilical points.

$(k_1 \neq k_2)$ principal curvature

coordinate curve are line of curvature $(f=0, F=0)$

(p.232) $\because$ $\Gamma_{11}^2 = \frac{-E_v}{2G}$, $\Gamma_{12}^1 = \frac{E_v}{2E}$

$\Gamma_{22}^1 =$, $\Gamma_{12}^2 =$

$e_v = e\Gamma_{12}^1 - g\Gamma_{11}^2 = \frac{E_v}{2}\left(\frac{e}{E} + \frac{g}{G}\right) = \frac{E_v}{2}(k_1 + k_2)$

$-g_u = -e\Gamma_{22}^1 + g\Gamma_{12}^2 = \frac{G_u}{2}\left(\frac{e}{E} + \frac{g}{G}\right) = \frac{G_u}{2}(k_1 + k_2)$

recall: $k_1 = \frac{e}{E}$, $k_2 = \frac{g}{G}$

$$\begin{aligned} KF &= \\ KE &= \\ KG &= \end{aligned} \left(\begin{matrix} \Gamma_{ij}^k \\ \\ E, F, G \\ E_u, E_v, F_u \\ F_v, G_u, G_v \end{matrix} \right)$$

$F=0$, $K = \frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$ ☆

$$K = \frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial}{\partial v}\left(\frac{-F}{2\sqrt{EG-F^2}}E_u + \frac{E}{\sqrt{EG-F^2}}\left(F_u - \frac{1}{2}E_v\right)\right) - \frac{\partial}{\partial v}\left[\frac{-F}{2\sqrt{EG-F^2}}E_v + \frac{1}{2}\frac{(G_u)E}{2\sqrt{EG-F^2}}\right)\right]$$ ☆☆

- $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ surface of revolution.

$f>0$, $(f')^2 + (g')^2 = 1$ (by arc-length) $v \in (0, 2\pi)$ $u \in \mathbb{R}$

$X_u = (f'\cos v, f'\sin v, g')$ $E = 1$

$X_v = (-f\sin v, f\cos v, 0)$ $F = 0$

$G = f^2$

By ☆, $K = \frac{1}{\sqrt{G}}\left(\frac{\partial^2 G}{\partial u^2}\right) = -\frac{f''}{f}$

extremely important

Gauss Egregium

The Gauss Curvature of a surface is invariant by local isometry.

$\because$ ☆☆ holds if $\varphi = \bar{X} \circ X^{-1}$ is local isometry.

$X: U \to S$ $E = \bar{E}$

$\bar{X}: U \to \bar{S}$ $F = \bar{F}$ $\boxed{K = \bar{K}}$

$G = \bar{G}$

Example: Catenoid . helicoid

$c: X(u,v) = (a\cosh v\cos u,\ a\cosh v\sin u,\ av)$

$h: \bar{X}(u,v) = (a\sinh v\cos u,\ a\sinh v\sin u,\ au)$

$$E = \bar{E} = a^2\cosh^2 v$$
$$F = \bar{F} = 0 \qquad , \quad K = \frac{-1}{a^2\cosh^4 v}$$
$$\bar{G} = \bar{G} = a^2\cosh^2 v$$

$$K = \frac{1}{\Delta}\left(\frac{\partial}{\partial v}\left(\frac{F E_u}{2E\Delta} + \frac{1}{\Delta}\left(F_u - \frac{E_v}{2}\right)\right) - \frac{\partial}{\partial u}\left(\frac{F E_v}{2E\Delta} + \frac{G_u}{2\Delta}\right)\right] \quad — ☆☆$$

$$\Delta = \sqrt{EG - F^2}$$

$$F = 0,\quad K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) \quad — ★$$

Gauss Egrium Theorem (Gauss remarkable Theorem)

The Gauss Curvature of a surface is preserved by local isometries.

$$\varphi : S_1 \xrightarrow[\text{isometry}]{\text{local}} S_2$$

$$I_p(w) = I_{\varphi(p)}(d\varphi(w)),\quad w = \alpha',\ w \in T_p(S)$$

$$\Rightarrow K(S_1) = K(S_2)$$

Example: ① $\mathbb{P}$ = flat plane cylinder, $K(\mathbb{P}) = K(C) = 0$.

$E_1 = E_2$
$F_1 = F_2$
$G_1 = G_2$

② catenoid $X(u,v) = (a\cosh v\cos u,\ a\cosh v\sin u,\ av)$
helicoid

$K(c) = K(h)$

local isometry

E_1
$F_1 = 0$
$G_1 = a^2\cosh^2 v$

By ★, $E = a^2\cosh^2 v$, $E_v = 2a^2\sinh v\cosh v$
$G = a^2\cosh^2 v$
$G_u = 0$.

$$K(c) = \frac{-1}{2(a^2\cosh^2 v)}\left(\left(\frac{2a^2\sinh v\cosh v}{a^2\cosh^2 v}\right)_v + 0\right)$$

$$\left(= \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)\right) = \frac{-1}{a^2\cosh^4 v} = K(h)$$

However, the converse to the Egregium Theorem is not true.

Counterexample :

S_1: $X(u,v) = (u\cos v, u\sin v, \log u)$ $\quad u>0, \; v\in(0,2\pi)$ surface of revolution

S_2: $\bar{X}(u,v) = (u\cos v, u\sin v, v)$ helicoid

$X_u(u,v) = (\cos v, \sin v, \frac{1}{u})$

$X_v(u,v) = (-u\sin v, u\cos v, 0)$

$E = \langle X_u, X_u\rangle = \frac{u^2+1}{u}$

$F = \langle X_u, X_v\rangle = 0$

$G = \langle X_v, X_v\rangle = u^2$.

$$\frac{(\sqrt{1+u^2})^2 - 2u\left(\frac{2u}{2\sqrt{1+u^2}}\right)}{(1+u^2)}$$

$$K(S_1) \underset{\text{by} \bigstar}{=\!=} \frac{1}{2\sqrt{1+u^2}}\left(\left(\frac{0}{\sqrt{1+u^2}}\right)_v + \underline{\left(\frac{2u}{\sqrt{1+u^2}}\right)_u}\right) = \frac{-1}{(1+u^2)^2}$$

$\bar{X}_u(u,v) = (\cos v, \sin v, 0)$

$\bar{X}_v(u,v) = (-u\sin v, u\cos v, 1)$

※ Change u, v

$\Rightarrow \bar{G} = 1, \bar{E} = u^2+1$, $G=\bar{G}$, $E=\bar{E}$ only at one point

$\Rightarrow$ still not local isometry.

$\bar{E} = \langle \bar{X}_u, \bar{X}_u\rangle = 1$

$\bar{F} = \langle \bar{X}_u, \bar{X}_v\rangle = 0$

$\bar{G} = \langle \bar{X}_v, \bar{X}_v\rangle = u^2+1$

$$K(S_2) = \frac{-1}{2\sqrt{u^2+1}}\left(\left(\frac{0}{\sqrt{1+u^2}}\right)_v + \left(\frac{2u}{\sqrt{1+u^2}}\right)_u\right) = \frac{-1}{(1+u^2)^2}$$

by $\bigstar$ ($\because F=0$)

$K(S_1) = K(S_2)$ but $E \neq \bar{E}$, $F \neq \bar{F}$

$$N = \frac{X_u \wedge X_v}{|X_u \wedge X_v|} = \frac{(-\cos v, \sin v, u)}{\sqrt{1+u^2}}, \quad K = \frac{eg-f^2}{EG-F^2}.$$

<u>Theorem</u> : (Bonnet Fundamental Theorem of surface)

Let E, F, G, e, f, g be differential functions defined on an open set $V \subseteq \mathbb{R}^2$ with $E > 0$, $G > 0$ and $EG - F^2 > 0$

Assume that they satisfy the Gauss-formulas and codazzi - Mainardi equations. Then $\forall\, q \in V$, $\exists$ a nbhd $U \subseteq V$ of q and a diffeomorphism $X : U \to \mathbb{R}^3$ s.t $X(U) = S$ be a parametrization regular surface with the 1^{st} fundamental form and 2^{nd} fundamental form given by E, F, G and e, f, g respectively. Moreover, if U is connected and $\bar{X} : U \to \mathbb{R}^3$ is another parametrization surface with the same property, then $\bar{X}(U) = S$ can be obtained by applying a rigid motion of $\mathbb{R}^3$ to S.

<u>Recall</u> : Fundamental Theorem of curves.

κ, τ

$$\begin{cases} t' = \kappa n \\ n' = -\kappa t - \tau b \\ b' = \tau n \end{cases}$$

$E = (\xi_1, \xi_2, \xi_3)$

$F = (\xi_4, \xi_5, \xi_6)$ $\qquad$ ⊙ $X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + eN$

$\vdots$

$G = (\xi_7, \xi_8, \xi_9)$

We will have a system of P.D.E

$(\xi_1)_u = f_1(u, v, \xi_1, \dots, \xi_9)$

$(\xi_2)_u = f_2(u, v, \xi_1, \dots, \xi_9)$

<u>refer</u> : end of chapter 4

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<u>Example</u> : $X : U \to \mathbb{R}^3$ with $\triangle \begin{cases} E = 1 & e = \cos^2 u \\ F = 0 & f = 0 \\ G = \cos^2 u & g = 1 \end{cases}$

Check : is there a regular surface s.t $\triangle$ holds.

① $K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) = \frac{-1}{2\cos u}\left(0 + \left(\frac{-2\cos u \sin u}{\cos u}\right)_u\right)$

$= \frac{\cos u}{\cos u} = 1.$

② check Christoffel symbol.

$\Gamma^1_{11} = \frac{1}{GE}\cdot\frac{1}{2}[GE_u] = \frac{E_u}{2E} = 0$

$\Gamma^2_{11} = \frac{1}{GE}\cdot\frac{-1}{2}EE_v = \frac{-E_v}{2G} = 0$

$\Gamma^1_{12} = \frac{1}{GE}\cdot\frac{1}{2}GE_v = \frac{E_v}{2E} = 0$

$\Gamma^2_{12} = \frac{1}{GE}\cdot\frac{1}{2}\cdot EG_u = \frac{G_u}{2G} = -\tan u$

$\Gamma^1_{22} = \frac{1}{GE}\cdot\frac{-1}{2}GG_u = \frac{-G_u}{2E} = \cos u \sin u$

$\Gamma^2_{22} = \frac{1}{GE}\cdot\frac{1}{2}\cdot EG_v = \frac{G_v}{2G} = 0$

recall :

$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} \frac{E_u}{2} \\ F_u - \frac{E_v}{2} \end{pmatrix}$

$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \begin{pmatrix} \frac{E_v}{2} \\ \frac{G_v}{2} \end{pmatrix}$

$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} F_v - \frac{G_u}{2} \\ \frac{G_v}{2} \end{pmatrix}$

③ Check Gauss Formula (equation)

$-EK = (\Gamma^2_{12})_u + \Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})^2 - (\Gamma^2_{11})_v - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}$ — *

$LHS* = -1 = -EK = -(+1)(+1)$

$RHS* = 0 + \underbrace{(-\tan u)_u}_{(\Gamma^2_{12})_u} + (-\tan u)^2 + 0 + 0.$

$= -\sec^2 u + \tan^2 u = -1$

④ Check Mainardi-Codazzi equation.

∴ Need Check

$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}$ — (A_1)

$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^2_{12}) - g\Gamma^2_{12}$ — (A_2)

In (A_1), $0 = 0 + 0 + 0$ ✓

$0 = \cos^2 u(\sin u \cos u) + 0 - 1(-\tan u)$

$= \cos^3 u \sin u + \tan u$

$= \tan u(\cos^4 u + 1)$ $\quad \cos^4 u + 1 = 0$ impossible

∴ Mainardi - codazzi ⊗

Bonnet : $\left.\begin{matrix} E, F, G \\ e, f, g \end{matrix}\right\}$

$\Gamma_{ij}^{k} = C(E, F, G, E_v, E_u, F_u, F_v, G_u, G_v)$

∴ By fundamental-Theorem of surface, there is no such surface with 1^{st} f.f E, F, G, 2^{nd} f.f e, f, g.

4-4 Parallel transport geodesic

Def: Let w be a differentiable vector field on $U \subseteq S$. (S is a regular surface) and $p \in U$

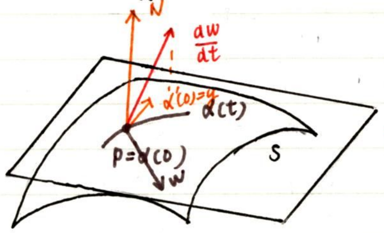
Let $y \in T_p(S)$, consider a parametrized curve $\alpha : (-\varepsilon, \varepsilon) \to U$ with $\alpha(0) = p$, $\alpha'(0) = y$

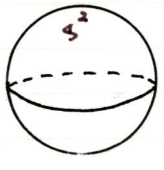
Let $w(t)$, $t \in (-\varepsilon, \varepsilon)$ be the restriction of vector field w on α. Consider the projection of $\frac{dw}{dt}(0)$ onto tangent space $T_p(S)$ and write this as $\frac{Dw}{dt}(0)$ (or $D_y w(p)$ or $\nabla_y w$)

We called $\frac{Dw}{dt}(0)$ the covariant derivative of w. w.r to y

$\parallel$

a vector obtained by $\frac{dw}{dt}$ normal projection.





$$\frac{Dw}{dt}(0) = \left(\frac{dw}{dt}\right)^T$$

$w(u,v) = a(u,v)X_u + b(u,v)X_v$

$\alpha(t) = X(u(t), v(t))$

$\alpha(0) = X(u(0), v(0)) = p$

$\alpha'(t) = X_u u' + X_v v' \Big|_{t=0} \in T_p(S)$

$= y$

$$\frac{dw}{dt} = \frac{d}{dt}(aX_u + bX_v) = a'X_u + a(X_{uu} \cdot u' + X_{uv}v') + b'X_v + b(X_{vu}u' + X_{vv}v')$$

By ⓪, $\frac{dw}{dt} = a'X_u + \left(a\left((\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + eN)u' + (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN)v'\right)\right)$

$+ b'X_v + b\left((\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN)u' + (\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + gN)v'\right)$

$\{X_u, X_v\}$

$\because \frac{DW}{dt}(0) = \left(\frac{dw}{dt}\right)^T\Big|_{t=0} = a'X_u + a\left((\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v)u' + (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v)v'\right)$

covariant derivative

$+ b'X_v + b(\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v)u' + b(\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v)v'$ — 回

<u>Example</u>: $\mathbb{P}$ <u>plane</u>

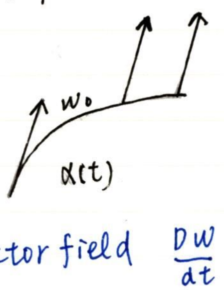
$\left.\begin{array}{l} E=1 \\ F=0 \\ G=1 \end{array}\right\} \Gamma_{ij}^k = 0.$

Look at $\frac{DW}{dt}(0) = a'X_u + b'X_v.$

<u>Def</u>:

A regular vector field w along a regular parametrized curve.
$\alpha: I \to S$ is said to be parallel if $\frac{DW}{dt} = 0 \quad \forall t \in I$

<u>prop</u>: Let $\alpha: I \to S$ be a parametrized curve in S.
Let $W_0 \in T_{\alpha(0)}S$, $t_0 \in I$
Then $\exists!$ parallel vector field $w(t)$ along $\alpha(t)$ with $w(t_0) = w_0$.



$\because$ parallel vector field $\frac{DW}{dt} = 0$

$\left.\text{回} = 0\right\} \Rightarrow a'X_u + a(\cdots) + b'X_v + b(\cdots) = 0.$

& $w(t_0) = w_0$.

X_u: $a' + a(\Gamma_{11}^1 u' + \Gamma_{12}^1 \cdots v') + b(\Gamma_{12}^1 u' + \Gamma_{22}^1 \cdots)$

X_v: $b' + a(\Gamma_{11}^2 X_v u' + \Gamma_{12}^2 X_v v') + b(\Gamma_{12}^2 X_v u' + \Gamma_{22}^2 X_v v')$

By O.D.E Theorey, the existence and uniqueness hold true.

$\therefore \exists$ parallel vector fields along a parametrized curve $\alpha(t)$ and they are completely determined by their value at t_0.

prop. Let V and W be two parallel vector fields ($\frac{DV}{dt}=0$, $\frac{DW}{dt}=0$) along $\alpha : I \to S$ (a parametrized curve). Then $\langle v, w \rangle = \text{const.}$ $= |v||w|\cos\theta$

In particular, $|v|$ and $|w|$ are const. and the angle between v and w is const.

$$\frac{d}{dt}\langle v, w\rangle = \langle \frac{dv}{dt}, w\rangle + \langle v, \frac{dw}{dt}\rangle$$

→ Tangent Component of $\frac{dv}{dt}$

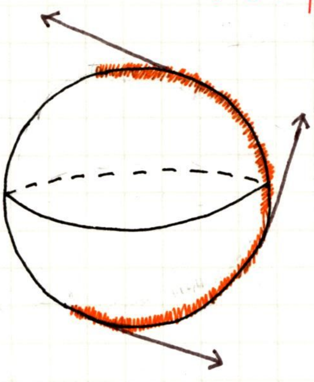
$$= \langle \left(\frac{dv}{dt}\right)^N + \left(\frac{dv}{dt}\right)^T, w\rangle + \langle v, \left(\frac{dw}{dt}\right)^N + \left(\frac{dw}{dt}\right)^T\rangle$$

↳ normal component of $\frac{dv}{dt}$

$$= \langle \left(\frac{dv}{dt}\right)^T, w\rangle + \langle V, \left(\frac{dw}{dt}\right)^T\rangle$$

$$= \langle \frac{DV}{dt}, w\rangle + \langle v, \frac{DW}{dt}\rangle$$

$= 0$ ($\because$ parallel v.f $\frac{DV}{dt} = 0 = \frac{DW}{dt}$)



meridian is a great circle

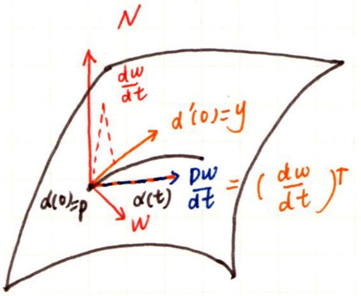
the tangent vector field w of a meridian (parametrized by arc-length) is a parallel field on S^2.

$W = \alpha'$

$w' = \alpha'' \perp S^2$

$\frac{dw}{dt} \perp S^2$, $\frac{DW}{dt} = 0$.

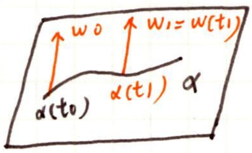
Covariant derivative



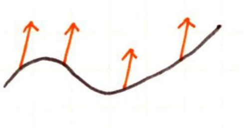
$\left.\frac{Dw}{dt}\right|_{t_0}=0$, w is a parallel.

Def: Let $\alpha: I \to S$ be parametrized curve on S and $w_0 \in T_{\alpha(0)}(S)$
$\uparrow$ vector field

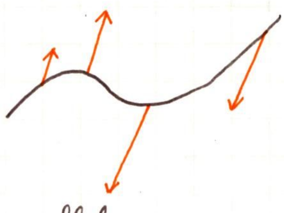
Let w be the parallel vector field along α with $w(t_0)=w_0$. The vector $w_1 = w(t_1)$, $t_1 \in I$ is called the parallel transport of w_0 along α at the point $\alpha(t_1)$



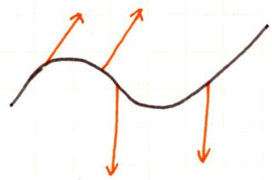
$w_0 // w_1$ and Preserves length



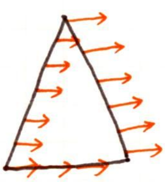
parallel transport



parallel



not parallel



Note that

① Let $P_\alpha: T_p(S) \to T_q(S)$ where $\begin{matrix}\alpha(0)=p \\ \alpha(1)=q\end{matrix}$ be a parallel transport of w along α, then P_α is isometry

② Parallel transport is intrinsic depends on the tangent space and is independent of the parametrization of α.

$\because$ If $\beta: J \to S$ be another parametrization of the curve.

Let $\eta \in J$

$\because \frac{dt}{d\eta} \neq 0$

$$\frac{Dw}{d\eta} = \left(\frac{dw}{d\eta}\right)^T = \left(\frac{dw}{dt}\frac{dt}{d\eta}\right)^T = \left(\frac{Dw}{dt}\right)\left(\frac{dt}{d\eta}\right)^T$$

$\boxed{\text{w(t) is parallel (transport) iff w(}\eta\text{) is parallel transport}}$

③ A parallel transport can be defined on <u>piecewise regular curve</u>

$\alpha: [a,b] \to S$

$a < t_1 < t_2 < t_3 < \cdots < t_k < b$

$\alpha \big| [t_i, t_{i+1}]$

<u>Def</u>: A parametrized (regular) curve $\alpha: I \to S$ is a <u>parametrized geodesic</u> if $\frac{D\alpha'}{dt} = 0$, $\forall t \in I$.

<u>Def</u>: A curve C is a geodesic if $\forall p \in C$ (curve), $\exists$ a parametrization $\alpha(s)$ of a Nbd of p by arc-length which is a parametrized geodesic. $\boxed{\frac{D(\alpha'(s))}{ds} = 0}$

<u>observe</u>: If $\alpha(t)$ is a parametrized geodesic, then $\alpha'(t)$ is a parallel vector field ($\because \frac{D\alpha'(t)}{dt} = 0$)

Hence, $|\alpha'(t)| = \text{constant} = \frac{1}{a}$ $a \neq 0$, const.

One can adjust the parametrization by setting $s = \frac{t}{a}$ and $\alpha(s)$ is parametrized by arc-length say β

$\frac{D(\beta'(s))}{ds} = 0$. $\beta(s) = \alpha(as)$.

$$\frac{d\beta}{ds} = \frac{d\beta}{dt}\frac{dt}{ds}$$

$$\left|\frac{d\beta}{ds}\right| = |\alpha'||a| = 1.$$

$$\frac{D(\beta'(s))}{ds} = \left(\frac{d\beta'}{dt}\cdot\frac{dt}{ds}\right)^T = (\alpha''\, a^2)^T = \frac{1}{|\alpha'|^2}\frac{D\alpha'}{dt}$$

$\therefore$ if α is the curve parametrized by arc-length

$|\alpha'|^2 = 1$, $\dfrac{D(\beta'(s))}{ds} = \left(\dfrac{D\alpha'}{dt}\right)$

Def : Let w be differentiable unit vector field along $\alpha : I \to S$ on a oriented surface $\frac{dw}{dt} \perp w$ (But it is not necessary $\perp S$)

$\dfrac{Dw}{dt} = \lambda(t)(N \wedge W) = \left[\dfrac{Dw}{dt}\right](N \wedge W)$, where $\left[\frac{Dw}{dt}\right]$ is called algebra value of $\frac{dw}{dt}$

sign of $\left[\frac{Dw}{dt}\right]$ depends on the orientation of S.

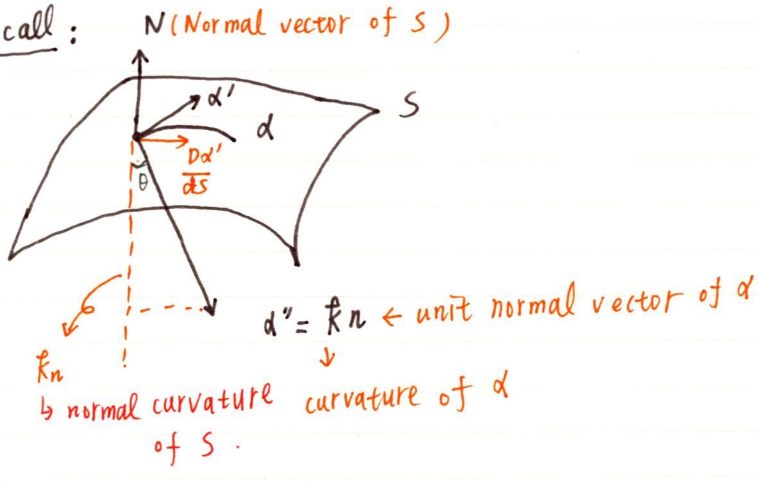
If $w = \alpha'(s)$ (α is a parametrized curve by arc-length)

$\left[\frac{Dw}{dt}\right] = \lambda(t) = k_g$ is called the geodesic curvature of α

$\dfrac{D\alpha'(s)}{ds} = 0 = k_g(N \wedge \alpha')$ $\therefore$ $k_g = 0$

α is geodesic, $k_g = 0$.

recall :



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$k_n = k\cos\theta$

$k_g = \pm k\sin\theta$.

$\because \alpha' \perp N$

$\alpha' \perp N \wedge \alpha'$

$\alpha'' \perp \alpha'$

$\alpha'' = AN + BN\wedge\alpha'$

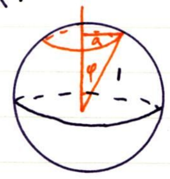
$k^2 = k_n^2 + k_g^2$ --- ⊛

· In fact, $\alpha'' = k_n N + k_g N\wedge\alpha'$

$\alpha''\cdot N = k_n$

$\alpha''\cdot N\wedge\alpha' = k_g$

Ex:



$S^2(1)$

$a = 1\cdot\sin\varphi = \sin\varphi$

↓

半徑 of sphere

By ⊛, $k^2 = k_n^2 + k_g^2$

在小圓's curvature $= \frac{1}{a} = \frac{1}{\sin\varphi}$ and k_n (normal curvature) = 1.

$\frac{1}{\sin^2\varphi} = 1 + k_g^2 \Rightarrow \boxed{k_g^2} = \cot^2\varphi = \frac{\cos^2\varphi}{\sin^2\varphi}$

$k_g \to \infty$ if $\sin\varphi \to 0$.

$\{\alpha', N, N\wedge\alpha'\}$ forms a basis.

α is a regular curve

$\alpha'' = \underbrace{k_n}_{\text{normal curvature}} N + \underbrace{k_g}_{\text{geodesic curvature}} N\wedge\alpha'$

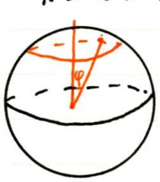
$\parallel$

$k\cdot n$

$k_n = \alpha''\cdot N$

$k_g = \alpha''\cdot N\wedge\alpha'$

$k^2 = \langle\alpha'', \alpha''\rangle = \langle k_n N + k_g N\wedge\alpha', k_n N + k_g N\wedge\alpha'\rangle \Leftrightarrow k^2 = k_n^2 + k_g^2$



$k_g^2 = \mathrm{cotan}^2\varphi$

<u>Example</u>: compute the normal curvature & geodesic curvature.

if the circle $\alpha(t) = (\cos t, \sin t, 1)$ on the elliptic Paralobic

$X(u,v) = (u, v, u^2+v^2)$

<u>sol</u>: $\alpha'(t) = (-\sin t, \cos t, 0)$ $\qquad |\alpha'(t)| = 1$ arc-length

$\alpha''(t) = (-\cos t, -\sin t, 0)$

($\because \cos^2 t + \sin^2 t = 1$, $\alpha(t)$ is contained in S)

$X_u = (1, 0, 2u)$, $X_u \wedge X_v = (-2u, -2v, 1)$

$X_v = (0, 1, 2v)$ $\quad |X_u \wedge X_v| = \sqrt{4u^2+4v^2+1}$

$$N = \frac{X_u \wedge X_v}{|X_u \wedge X_v|} = \frac{(-2u, -2v, 1)}{\sqrt{4u^2+4v^2+1}}$$

restriction of N on α (i.e $u = \cos t$, $v = \sin t$)

$N = \frac{1}{\sqrt{5}}(-2u, -2v, 1)$

$$N \wedge \alpha' = \begin{vmatrix} i & j & k \\ \frac{-2u}{\sqrt{5}} & \frac{-2v}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\sin t & \cos t & 0 \end{vmatrix} = \left(-\frac{1}{\sqrt{5}}\cos t, -\frac{1}{\sqrt{5}}\sin t, -\frac{2}{\sqrt{5}}\right)$$

$k_n = \langle \alpha'', N \rangle = \frac{2}{\sqrt{5}}$

$k_g = \langle \alpha'', N \wedge \alpha' \rangle = \frac{1}{\sqrt{5}}$

<u>Lemma</u>: Let a and b be differentiable functions & $a^2+b^2=1$ and at φ be such that $a(t_0) = \cos\varphi_0$, $b(t_0) = \sin\varphi_0$

Then the differentiable function defined by $\varphi(t) = \varphi_0 + \int_t^{t_0} (ab' - a'b)\,dt$

satisfies $\begin{cases} a(t) = \cos\varphi(t) \\ b(t) = \sin\varphi(t) \\ \varphi(t_0) = \varphi_0 \end{cases}$, $\forall t \in I$

"<u>pf</u>": Let $f(t) = (a - \cos\varphi(t))^2 + (b - \sin\varphi(t))^2$

Hope $f(t) = 0 \ \forall t$

$f(t) = a^2 - 2a\cos\varphi(t) + \cos^2\varphi(t) + b^2 - 2b\sin\varphi(t) + \sin^2\varphi(t)$

$= 2(1 - a\cos\varphi(t) - b\sin\varphi(t))$

$f'(t) = 2[(-a'\cos\varphi(t) - b'\sin\varphi(t) + a\sin\varphi(t)\varphi' - b\cos\varphi\,\varphi'))$

$$f'(t) = 2\left[-a'\cos\varphi + a\sin\varphi\,(ab'-a'b) - b'\sin\varphi - b\cos\varphi\,(ab'-a'b)\right]$$

$$= 2\left[-a'\cos\varphi + a^2b'\sin\varphi - aa'b\sin\varphi - b'\sin\varphi - abb'\cos\varphi + a'b^2\cos\varphi\right]$$

$\because\ a^2+b^2=1 \ \therefore\ 2aa'+2bb'=0 \iff aa'=-bb'$

$$= 2\left[-a'\cos\varphi - b'\sin\varphi + a^2b'\sin\varphi + b^2b'\sin\varphi + a'b^2\cos\varphi + a'a^2\cos\varphi\right]$$

$= 0 \quad \therefore\ f'(t)=0 \Rightarrow f(t)$ const. $\Rightarrow$ $a=\cos\varphi$, $b=\sin\varphi$

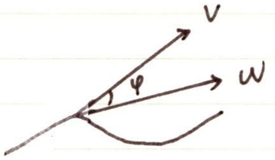
<u>Lemma2</u>: Let V and w be two differentiable vector fields along a curve $\alpha: I \to S$ with $|v|=1$, $|w|=1$, $\forall t \in I$.

Then $\left[\frac{Dw}{dt}\right] - \left[\frac{DV}{dt}\right] = \frac{d\varphi}{dt}$

φ is one of differentiable determination of the angle from v to w.

H.W

$$\varphi = \varphi_0 + \int_{t_0}^{t} (ab'-a'b)\,dt$$



If v is a parallel vector field $\frac{DV}{dt}=0$.

and $w=\alpha'$

$$\left[\frac{Dw}{dt}\right] = \left[\frac{D\alpha'}{dt}\right] = k_g = \frac{d\varphi}{dt}$$

<u>Note that</u>: Let $\alpha: I \to S$ parametrized by arc-length

Let $V(s)$ be a parallel vector field

$\therefore\ \frac{DV(s)}{ds} = 0$.

take $w=\alpha'(s)$ $\therefore$ $k_g = \left[\frac{D\alpha'(s)}{ds}\right] = \left[\frac{Dw}{dt}\right] = \frac{d\varphi}{dt}$

geodesic curvature = rate of change of the angle that the tangent to α makes with a parallel vector field V.

<u>Prop</u>: Let $X(u,v)$ be a <u>orthogonal Parametrization</u> ($F=0$) of a Nbhd of an oriented surface S, and $w(t)$ be differentiable field of unit vector ($|w|=1$) along the curve $\alpha = X(u(t), v(t))$

Then $\left[\frac{Dw}{dt}\right] = \underbrace{\frac{1}{2\sqrt{EG}}\left[G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right]}_{\left[\frac{DV}{dt}\right]} + \frac{d\varphi}{dt}$

φ = angle from w to X_u in the given orientation.

If use Lemma 2, $w = e_1 \wedge e_2$, $v = e_1 \Rightarrow \left[\frac{De_1}{dt}\right] =$

pf: X is orthogonal

$e_1 = \frac{X_u}{\sqrt{E}}$, $e_2 = \frac{X_v}{\sqrt{G}}$, $|e_1| = 1$, $|e_2| = 1$

Let $\{e_1, e_2\}$ form a basis for $T(S)$

$N = e_1 \wedge e_2$

Let α be the curve parametrization by arc-length

$\alpha' = \cos\varphi\, e_1 + \sin\varphi\, e_2 \in T(S)$, $\varphi =$ angle between α' to e_1

Let $\eta = N \wedge \alpha'$, $\eta \perp \alpha'$

$\eta = -\sin\varphi\, e_1 + \cos\varphi\, e_2$

$\alpha'' = (-\sin\varphi\, e_1 + \cos\varphi\, e_2)\varphi' + \cos\varphi\,(e_1)' + \sin\varphi\,(e_2)'$

$= \eta\varphi' + \cos\varphi\,(e_1)' + \sin\varphi\,(e_2)'$

$\left[\frac{Dw}{dt}\right] = \langle \alpha'', \eta \rangle = \langle \eta\varphi' + \cos\varphi\,(e_1)' + \sin\varphi\,(e_2'), -\sin\varphi\, e_1 + \cos\varphi\, e_2 \rangle$

$= \varphi' - \cos\varphi\sin\varphi\,\overset{0}{(e_1)'e_1} + \cos^2\varphi\,(e_1)'e_2 - \sin^2\varphi\,(e_2')e_1 + \sin\varphi\cos\varphi\,\overset{0}{(e_2')e_2}$

$|e_1| = 1 \Rightarrow \langle e_1, e_1' \rangle = 0$.

$|e_2| = 1 \Rightarrow \langle e_2, e_2' \rangle = 0$

$\langle e_1, e_2 \rangle = 0 \Rightarrow \langle e_1, e_2' \rangle = -\langle e_1', e_2 \rangle$

$\Rightarrow \langle \alpha'', \eta \rangle = \varphi' + \langle e_1', e_2 \rangle$

$\because e_1 = \frac{X_u}{\sqrt{E}}$ $\therefore e_1' = \underbrace{(e_{1,u})}_{\frac{\partial e_1}{\partial u}} u' + \underbrace{(e_{1,v})}_{\frac{\partial e_1}{\partial v}} v' = \left(\frac{X_u}{\sqrt{E}}\right)_u u' + \left(\frac{X_u}{\sqrt{E}}\right)_v v'$

$\langle e_1', e_2 \rangle = \left\langle \left(\frac{X_u}{\sqrt{E}}\right)_u u' + \left(\frac{X_u}{\sqrt{E}}\right)_v v', \frac{X_v}{\sqrt{G}} \right\rangle$

$= \left\langle \left(\frac{X_{uu}\sqrt{E} - X_u\left(\frac{E_u}{2\sqrt{E}}\right)}{E}\right) u' + \frac{X_{uv}\sqrt{E} - X_u\left(\frac{E_v}{2\sqrt{E}}\right)}{E}, \frac{X_v}{\sqrt{G}} \right\rangle$

$= \frac{1}{\sqrt{EG}} \underbrace{\langle X_{uu}, X_v \rangle}_{\substack{\langle X_u, X_v\rangle_u - \frac{1}{2}\langle X_u, X_u\rangle_v \\ = F_u - \frac{1}{2}E_v}} u' + \frac{1}{\sqrt{EG}} \underbrace{\langle X_{uv}, X_v \rangle}_{\substack{\frac{1}{2}\langle X_v, X_v\rangle_u \\ = \frac{1}{2}G_u}} v' = \frac{1}{2\sqrt{EG}}\left(G_u \frac{dv}{dt} - E_v \frac{du}{dt}\right)$

$\Rightarrow \left[\frac{DW}{dt}\right] = \langle \alpha'', \eta \rangle = \langle e_1', e_2 \rangle + \varphi'$

$= \frac{1}{2\sqrt{EG}}\left(G_u \cdot \frac{dv}{dt} - E_v \frac{du}{dt}\right) + \varphi'$

<u>recall</u>: $X(u,v)$: an orthogonal parametrization.

$w \in C^\infty$, <u>vector field</u>.

$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u v' - E_v u'\right) + \frac{d\varphi}{dt}$, φ = angle between X_u and w.

$\alpha : I \to S$ parametrized curve by <u>arc-length</u>

$\alpha' = w$ $\quad \alpha' = X_u u' + X_v v'$

$k_g = \left[\frac{D\alpha'}{ds}\right] = \frac{1}{2\sqrt{EG}}\left(G_u v' - E_v u'\right) + \frac{d\varphi}{ds} = (k_g)_1 \cos\varphi + (k_g)_2 \sin\varphi + \frac{d\varphi}{ds}$.

$(k_g)_1$: v = const. $\quad (k_g)_2$: u = const.

k_g geodesic curvature.

Look at <u>coordinate curves</u>.

<u>case 1</u>: $u(s) = u(s)$, $v = v_0$ const. $u' = u'$, $v' = 0$.

$(k_g)_1 \Rightarrow$ geodesic curvature of coordinate curve v = const.

$(k_g)_1 = \frac{1}{2\sqrt{EG}}(-E_v u') + 0 = \frac{-E_v u'}{2\sqrt{EG}} = \frac{-E_v}{2\sqrt{G}E}$

$\left|\frac{dX(u,v_0)}{ds}\right| = 1$, $\left|\frac{\partial X}{\partial u}\frac{\partial u}{\partial s}\right| = 1$, $\left|\frac{\partial u}{\partial s}\right| = \frac{1}{\sqrt{E}}$

<u>case 2</u>: $u = u_0$, $v(s) = v(s)$

$u' = 0$, $v' = v'$

$(k_g)_2$ geodesic curvature of coordinate curve u = constant.

$1 = \left|\frac{dX(u_0,v)}{ds}\right| \Leftrightarrow \left|\frac{dv}{ds}\right| = \frac{1}{\sqrt{G}}$

$(k_g)_2 = \frac{1}{2\sqrt{EG}} G_u v' = \frac{G_u}{2G\sqrt{E}}$

$\cos\varphi = \langle \alpha', e_1 \rangle = \langle X_u u' + X_v v', \underbrace{\frac{X_u}{\sqrt{E}}}_{e_1} \rangle = \sqrt{E}\, u' + \cancel{\langle X_u, X_v \rangle}^{\,0} \frac{v'}{\sqrt{E}}$

$\sin\varphi = \langle \alpha', e_2 \rangle = \langle X_u u' + X_v v', \frac{X_v}{\sqrt{G}} \rangle = \sqrt{G}\, v'$

prop : (Liouville)

Let α be a parametrization by arc-length of Nbd of a point $p \in S$ of a regular oriented curve C on an oriented surface S.

Let $X(u,v)$ be an orthogonal parametrization of S at p. and $\varphi(s)$ be the angle from X_u to α'.

Then $k_g = (k_g)_1 \cos\varphi + (k_g)_2 \sin\varphi + \frac{d\varphi}{ds}$ $\quad [\frac{d\alpha'}{ds}] = \frac{1}{2\sqrt{EG}}(G_u v' - E_v u') + \frac{d\varphi}{ds}$.

where $(k_g)_1$ and $(k_g)_2$ are geodesic curvature of coordinate curves v = constant, u = const respectively.

※ $w \in C^\infty$, vector field

$$\frac{Dw}{dt} = 0 \text{ if } w \text{ is a parallel vector field}$$

$$\frac{Dw}{dt} = \left(\frac{dw}{dt}\right)^T$$

$$\boxed{w = aX_u + bX_v, \quad \alpha' = w, \quad a = u', \quad b = v'}$$

$$\frac{Dw}{dt} = \underbrace{\left[a' + a\Gamma_{11}^1 u' + b\Gamma_{12}^1 u' + b\Gamma_{22}^1 v' + a\Gamma_{12}^1 v'\right]}_{A} X_u +$$

$$+ \underbrace{\left[b' + a\Gamma_{11}^2 u' + a\Gamma_{12}^2 v' + b\Gamma_{22}^2 v' + b\Gamma_{12}^2 u'\right]}_{B} X_v$$

$\frac{Dw}{dt} = 0 \Leftrightarrow A = 0, B = 0$. — ☐ recall last time.

$\alpha : I \to S$ (surface) parametrized by arc-length.

$X : U \to V \subset S$ $\quad$ V is a nbhd of $\alpha(t)$, $t \in J$

$J \subset I$ s.t $\alpha(J) \subseteq V$ $\quad$ Tangent vector $\alpha'(s)$ $\forall s \in J$

$$\boxed{\alpha' = X_u u' + X_v v'}$$

$\frac{D\alpha'}{ds} = 0$ $\left(= \nabla_{\alpha'}\alpha' = 0\right)$ iff

$$\odot \begin{cases} u'' + \left(\Gamma_{11}^1 (u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 (v')^2\right) = 0 \\ v'' + \left(\Gamma_{11}^2 (u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 (v')^2\right) = 0 \end{cases}$$

$\odot$ are called geodesic equations.

- Geometry interpretation of geodesic.

Thm : A curve α on a surface S is a geodesic iff, for any part $\alpha(t) = X(u(t), v(t))$ of α contained in a coordinate Nbhd of S, the following equations are satisfied.

$\tilde{\circledcirc}$ $\begin{cases} \frac{d}{dt}(Eu'+Fv') = \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2) \\ \frac{d}{dt}(Fu'+Gv') = \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2) \end{cases}$

where $Eu'^2 + 2Fu'v' + Gv'^2$ is the 1^{st} fundamental form of X.

$\tilde{\circledcirc}$ are called geodesic equation $\circledcirc \sim \tilde{\circledcirc}$

(pf) Since $\{X_u, X_v\}$ is a basis of the tangent plane of X.

α is geodesic iff α'' perpendicular to X_u and X_v. i.e. $\underset{-①}{\langle \alpha'', X_u \rangle = 0}$, $\underset{-②}{\langle \alpha'', X_v \rangle = 0}$.

where $\alpha'' = \frac{d^2\alpha}{dt^2}$, $u' = \frac{du}{dt}$, $v' = \frac{dv}{dt}$.

$$\alpha' = \frac{d\alpha}{dt} = X_u \cdot u' + X_v v'$$

$$0 = \langle \alpha'', X_u \rangle = \langle \frac{d\alpha'}{dt}, X_u \rangle = \langle \frac{d}{dt}(X_u u' + X_v v'), X_u \rangle$$

$$= \frac{d}{dt}(\langle X_u u' + X_v v', X_u \rangle) - \langle X_u u' + X_v v', \frac{dX_u}{dt} \rangle$$

$$= \frac{d}{dt}(Eu'+Fv') - \langle X_u u' + X_v v', X_{uu} u' + X_{uv} v' \rangle$$

$$\Rightarrow \frac{d}{dt}(Eu'+Fv') = \langle X_u, X_{uu} \rangle u'^2 + \langle X_u, X_{uv} \rangle u'v' + \langle X_v, X_{uu} \rangle u'v' + \langle X_v, X_{uv} \rangle v'^2$$

$$= \frac{1}{2} E_u u'^2 + \frac{1}{2} E_v u'v' + (F_u - \frac{1}{2} E_v) u'v' + \frac{1}{2} G_u v'^2$$

$$= \frac{1}{2}[E_u u'^2 + 2F_u u'v' + G_u v'^2].$$

$$\langle X_u, X_v \rangle_u = \langle X_{uu}, X_v \rangle + \langle X_u, X_{uv} \rangle$$

HW – ②

$$0 = \langle \alpha'', X_v \rangle = \langle \frac{d\alpha'}{dt}, X_v \rangle = \frac{d}{dt}(\langle X_u u' + X_v v', X_v \rangle) - \langle X_u u' + X_v v', \frac{dX_v}{dt} \rangle$$

$$= \frac{d}{dt}(Fu'+Gv') - \langle X_u u' + X_v v', X_{uv} u' + X_{vv} v' \rangle$$

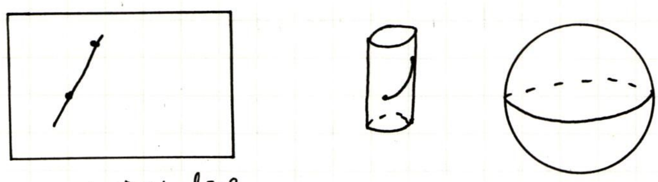
$$\langle X_u u' + X_v v', X_{uv} u' + X_{vv} v' \rangle =$$

$$= \langle X_u, X_{uv} \rangle u'^2 + (\langle X_{uv}, X_v \rangle + \langle X_u, X_{vv} \rangle) u'v' + \langle X_v, X_{vv} \rangle v'^2$$

$$= \frac{1}{2}[E_v u'^2 + 2F_v u'v' + G_v v'^2]$$

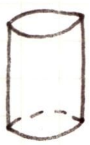
RmK: The geodesic equations ($\odot$ or $\tilde{\odot}$) are non-linear differential equations, and are usually difficult or impossible to slove explicity.

$E, F, G = 0$



straight line

Example :(Cylinder)



$X(u,v) = (\cos u, \sin u, v)$

$X_u = (-\sin u, \cos u, 0)$
$X_v = (\ 0\ ,\ 0, 1)$ $\rightarrow$ orthogonal

$E = \langle X_u, X_u \rangle = 1$
$F = \langle X_u, X_v \rangle = 0$
$G = \langle X_v, X_v \rangle = 1$

By geodesic equations($\tilde{\odot}$)

$E_u = 0,\ E_v = 0,\ F_u = 0,\ F_v = 0,\ G_u = 0,\ G_v = 0$

$\frac{d}{dt}(Eu') = 0$, $\frac{d}{dt}(u') = 0$ $\therefore$ u is linear

$\frac{d}{dt}(Gv') = 0$, $\frac{d}{dt}(v') = 0.$ v is linear.

$\Rightarrow$ $u = at+b$, $v = ct+d$, where a, b, c, d const.

If $a = 0$,

$\begin{cases} u = b \\ v = ct+d \end{cases}$ $\therefore$ geodesic is a <u>straight line</u>

If $a \neq 0$, $X(u,v) = (\cos(at+b), \sin(at+b), ct+d)$

$\begin{cases} u = at+b \\ v = ct+d \end{cases}$ $\therefore$ geodesic is a circular helix

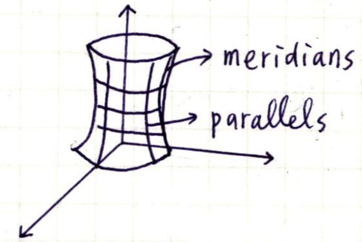
If by $\odot$, $\Gamma_{ij}^{k} = 0\ \forall\, i,j,k \Rightarrow \begin{cases} u'' = 0 \\ v'' = 0 \end{cases}$

Example : (Surface of revolution)

$$X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v)), \quad f(v)>0$$

$$X_u = (-f\sin u,\ f\cos u,\ 0)$$

$$X_v = (f'\cos u,\ f'\sin u,\ g')$$



$$E = \langle X_u, X_u\rangle = f^2$$

$$F = \langle X_u, X_v\rangle = 0$$

$$G = \langle X_v, X_v\rangle = f'^2 + g'^2$$

$E_u = 0$, $E_v = 2ff'$, $F_u = 0 = F_v$, $G_u = 0$, $G_v = 2(f'f + g'g)$

$$\Gamma^1_{11} = \frac{1}{2(EG-F^2)}\left[E_u G - 2F F_u + E_v F\right] = \frac{E_u}{2E} = 0.$$

$$\Gamma^2_{11} = \quad = \frac{-E_v}{2G} = \frac{-ff'}{f'^2+g'^2}$$

$$\Gamma^1_{12} = \quad = \frac{E_v}{2E} = \frac{f'}{f}$$

$$\Gamma^2_{12} = \quad = \frac{G_u}{2G} = 0$$

$$\Gamma^1_{22} = \quad = \frac{-G_u}{2E} = 0$$

$$\Gamma^2_{22} = \quad = \frac{G_v}{2G} = \frac{f'f'' + g'g''}{f'^2+g'^2}$$

plug Γ^k_{ij} into ⓪,

$$u'' + 2\Big(\underbrace{\frac{f'}{f}}_{\Gamma^1_{12}}\Big)u'v' = 0 \quad \text{—— ①}$$

$$v'' + \underbrace{\frac{-ff'}{f'^2+g'^2}}_{\Gamma^2_{11}} u'^2 + \underbrace{\frac{f'f''+g'g''}{f'^2+g'^2}}_{\Gamma^2_{22}} v'^2 = 0 \quad \text{—— ②}$$

$\boxed{\text{case 1:}}$ $u = u_0$, $v = v$

$u' = 0$

① o.K, $u' = 0$

② can be written as $v'' + \left(\dfrac{f'f''+g'g''}{f'^2+g'^2}\right)(v')^2 = 0.$

If α is a curve by arc-length

$$1 = |\alpha'|^2 = |\langle X_u u' + X_v v',\ X_u u' + X_v v'\rangle|$$

$$= E(u')^2 + 2F(u'v') + G(v')^2$$

Now $\begin{cases} u = u_0, & u' = 0, \\ v = v, & v' = v' \end{cases}$, $1 = E(0) + F(0)u' = G(v')^2$.

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$1=(v')^2\underbrace{(f'^2+g'^2)}_{G} \rightarrow (v')^2=\frac{1}{f'^2+g'^2}=\frac{1}{G}$

$2v'v''=\frac{-(f'^2+g'^2)'}{(f'^2+g'^2)^2}=\frac{-2(f'f''+g'g'')v'}{(f'^2+g'^2)^2}=\frac{-2(f'f''+g'g'')}{(f'^2+g'^2)}(v')^3.$

$\therefore$ geodesic $\quad u=u_0$
$v=v$

$\therefore X(u,v)=(f(v)\cos u_0,\ f(v)\sin u_0,\ v)$

All merdians are geodesic.

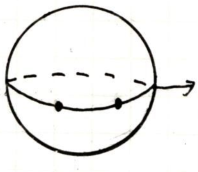
case 2 II: $\begin{cases} u=u & u'=u' \\ v=v_0, & v'=0 \end{cases}$

Now, $u'=u$, $v'=0$ (parallels) $\quad \begin{cases} u''=0 \\ \frac{-ff'}{f'^2+g'^2}(u')^2=0. \end{cases}$ $\quad u$ is linear

$f>0$, Hope $f'(u')^2=0$.

$1=E(u'^2)=f^2(u')^2$, $(u')^2=\frac{1}{f^2} \rightarrow \frac{f'(v_0)}{f^2(v_0)}=0.$

geodesic iff $f'(v_0)=0$, $X(u,v)=(f(v_0)\cos u,\ f(v_0)\sin u,\ g(v_0))$



$X(\theta,\varphi)=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,\sin\theta)$

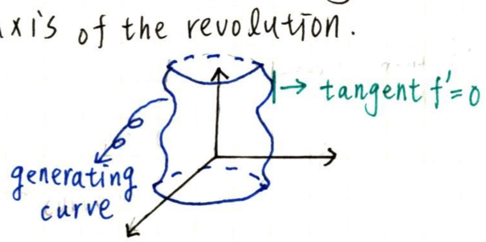
$f(\theta)=\cos\theta$, $g'(\varphi)=\sin\theta$

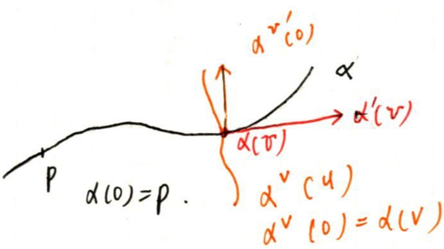
E, F, G (HW)

Fact: Curve α is contained in the intersection of S^2 with a plane passing through the original. Hence, in all cases, α is part of great circle.

Conclusion: (Surface of Revolution)

A necessary condition for a parallel of a surface of revolution to be geodesic is that such a parallel be generated by the part relation of a point of the generating curve where the tangent is parallel to the axis of the revolution.





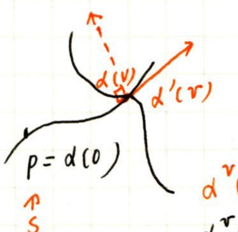
$X(u,v) = \alpha^v(u)$ geodesic Parametrization

u, v are called geodesic coordinates.

In geodesic coordinates, $E = 1$
$F = 0$
$G = \text{Free} = G(u,v)$

G is a smooth function s.t $G(0,v) = 1$
$G_u(0,v) = 0 \quad \forall (0,v) \in U$

- geodesic coordinates.



$\alpha^{v\prime}(u) \perp \alpha'(v)$

α is a geodesic (arc-length)

$\alpha^v(u)$ is a geodesic
$\alpha^v(0) = \alpha(v)$

$X(u,v) = \alpha^v(u)$, u, v are called geodesic coordinates.

$\exists\ U \subseteq \mathbb{R}^2$ containing $(0,0)$ s.t $X: U \to \mathbb{R}^3$ is a parametrization of S.

And 1^{st} fundamental form, $du^2 + G(u,v)dv^2$

$E = 1, F = 0, G = G(u,v)$ (is free) where G is a smooth function on U

s.t $G(0,v) = 1$
$G_u(0,v) = 0 \quad \forall (0,v) \in U$

※ $u = 0$, v is moving along α
$u = 0$, $v = 0$.
$\alpha'(0) \perp \alpha^{0\prime}(0)$

<u>Theorem</u>: Any point of a surface of constant Gaussian curvature is contained in a coordinate Nbhd that is isometric to an open set of a plane, a sphere or a pesudo-sphere

(i.e A surface with $K = 0$ S is locally isometric to a plane ($E = G = 1, F = 0$)
$K > 0$ sphere ($E = 1, F = 0, G = \cos^2\theta$)
$K < 0$ pesudo-sphere.)

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pf: $p\in S$, $K\equiv$ const.

Let $X(u,v)$ be a geodesic parametrization.

(i.e. u,v are geodesic coordinates) with $X(0,0)=p$.

∴ The first fundamental form $E\,du^2+2F\,du\,dv+G\,dv^2=du^2+G(u,v)dv^2$

$(E(u')^2+2F(u')(v')+G(v')^2)$

recall: p. 237.

$$K=\frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

∵ $E=1$, $G=G(u,v)$, ∴ $K=\frac{-1}{2\sqrt{G}}\left(\frac{G_u}{\sqrt{G}}\right)_u$

Let $\tilde{g}=\sqrt{G}$, $\tilde{g}_u=\frac{G_u}{2\sqrt{G}}$, $K=\frac{-1}{\tilde{g}}(\tilde{g}_u)_u=\frac{-1}{\tilde{g}}\frac{\partial^2\tilde{g}}{\partial u^2}$.

$$\begin{cases}\frac{\partial^2\tilde{g}}{\partial u^2}+K\tilde{g}=0.\\ \tilde{g}(0,v)=1\\ \tilde{g}_u(0,v)=0\end{cases}$$

case 1: $K=0$, $\frac{\partial^2\tilde{g}}{\partial u^2}=0\Rightarrow\tilde{g}=au+b$. ∴ $a=0$, $b=1$. $a,b\in C^\infty$

∴ $\tilde{g}=1$ ∴ $G=1$

$E=1, F=0, G=1$ ∴ locally isometric to a plane.

case 2: $K=1>0$.

$$\begin{cases}\tilde{g}_{uu}+\tilde{g}=0\\ \tilde{g}(0,v)=1\\ \tilde{g}_u(0,v)=0.\end{cases}$$

∴ $\tilde{g}=a\cos u+b\sin u$

$\tilde{g}(0,v)=1=a$

$\tilde{g}_u(0,v)=-a\sin(0)+b\cos(0)=b=0$

∴ $\tilde{g}=\cos u$ ∴ $G=\cos^2 u$.

1 st fundamental form is $du^2+\cos^2u\,dv^2$

∴ S is locally isometric to a sphere.

case 3 : $K<0$, $K=-1$

$$\begin{cases} \tilde{g}_{uu} - \tilde{g} = 0. \\ \tilde{g}(0,v) = 1 \\ \tilde{g}_u(0,v) = 0. \end{cases} \Rightarrow \tilde{g} = a\sinh u + b\cosh u$$

$\tilde{g}(0,v) = a\sinh(0) + b\cosh(0) \Rightarrow b = 1$

$\tilde{g}_u(0,v) = 0 \Rightarrow a = 0$.

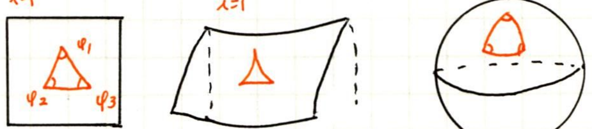
$E = 1, F = 0, G = \cosh^2 u$

1st fundamental , S is locally isometric to a pesudo-sphere (p.108)

★ Gauss-Bonnet Theorem

Consider geodesic triangle ($k_g = 0$)

$\sum_{i=1}^{3} \varphi_i = \pi$ $\quad$ $\sum_{i=1}^{3} \varphi_i < \pi$ $\quad$ $\sum_{i=1}^{3} \varphi_i > \pi$



φ_i = interion angles of geodesic triangle (each edge is geodesic)

Fact : $\sum_{i=1}^{3} \varphi_i - \pi = \iint_T K \, d\sigma$

$K = 0$, $\sum_{i=1}^{3} \varphi_i = \pi$

$K > 0$, $\sum_{i=1}^{3} \varphi_i > \pi$

$K < 0$, $\sum_{i=1}^{3} \varphi_i < \pi$

Def : α is a simple closed , piecewise regular parametrizd curve if

$\alpha : [0, l] \to S$

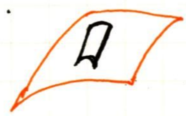
① $\alpha(0) = \alpha(l)$ closed.

② $t_1 \neq t_2$, $t_1, t_2 \in [0, l]$, $\alpha(t_1) \neq \alpha(t_2)$ (simple)

③ $\exists$ a sulidivision of $[0, l]$ given by $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = l$

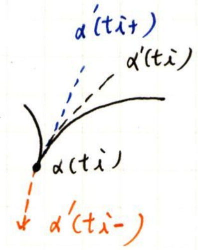
$\ni$ α is differentiable and regular in each $[t_i, t_{i+1}]$, $i = 0, \dots, k$

Def : The point $\alpha(t_i)$, $i = 0, \dots, k$ are closed vertices of α and trace of $\alpha([t_i, t_{i+1}])$ are called regular arcs of α , we call the trace of $\alpha([0,1])$ a simple , closed, piecewise regular curve.



2018.05.23 (4)

$$\lim_{\substack{t \to t_i \\ t < t_i}} \alpha'(t) = \alpha'(t_{i-}) \neq 0$$

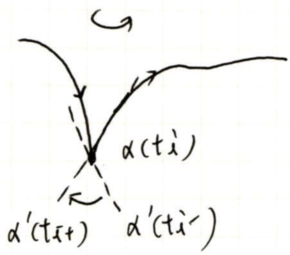


$$\lim_{\substack{t \to t_i \\ t > t_i}} \alpha'(t) = \alpha'(t_{i+}) \neq 0 .$$

By the condition of regularity, for each vertex $\alpha(t_i)$, there exist the limit from left (from right)

If S is oriented, Let $|\theta_i|$. $0 < |\theta_i| < \pi$ be the smallest determination of the angle from $\alpha'(t_{i-})$ to $\alpha'(t_{i+})$

If $|\theta_i| \neq \pi$, we give the θ_i the sign determined by $(\alpha'(t_{i-}), \alpha'(t_{i+}), N)$



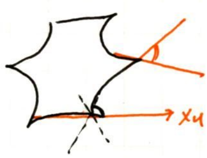
Def: Let $X: U \subseteq \mathbb{R}^2 \to S$ be a parametrization of S (oriented)

Assume that U is homeomorphic to an open disk in plane.

Let $\alpha: [0, l] \to X(U) \subseteq S$ be a simple, closed curve with vertices $\alpha(t_i)$ and external angle θ_i, $i = 0, 1, \ldots, k$

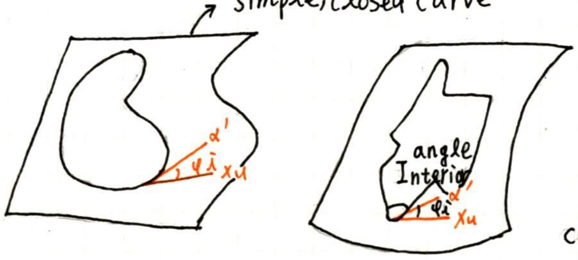
Let $\varphi_i : [t_i, t_{i+1}] \to \mathbb{R}$ differentiable function

+ it measures the positive angle from X_u to $\alpha'(t)$ on the regular arc $[t_i, t_{i+1}]$



— 5/24

simple, closed curve

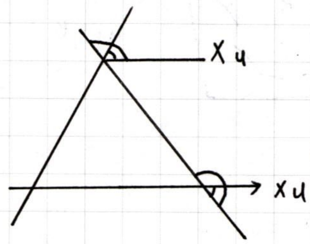


curve with corner

Let $\alpha: [0, l] \to X(U) \subseteq S$ be simple, closed, regular parametrized curve with vertice $\alpha(t_i)$ and external angles θ_i, $i = 0, 1, \dots, k$

<u>Proposition</u>. (Theorem of turning tangents)

$$\sum_{i=0}^{\infty} (\varphi_i(t_{i+1}) - \varphi_i(t_i)) + \sum_{i=1}^{k} \theta_i = \pm 2\pi,$$



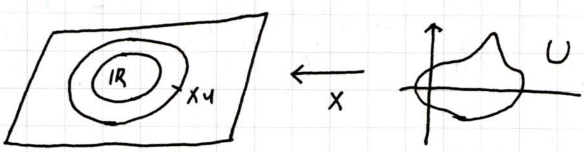
where $\varphi_i : [t_i, t_{i+1}] \to \mathbb{R}$. C^∞ function, which measure $t \in [t_i, t_{i+1}]$, the positive angle from X_u to α'.

$$\sum^{3} \varphi(t_{i+1}) - \varphi(t_i) = \text{cancel} = 0$$

$$\sum \theta_i = 2\pi$$

$f: S \to \mathbb{R}$ differentiable function

$X: U \subseteq \mathbb{R}^2 \to S^2$ $\quad$ $R \subset X(U)$ Nbhd region.



$$A(R) = \iint_{X^{-1}(R)} \sqrt{EG-F^2}\, du\, dv = \iint_{X^{-1}(R)} \sqrt{EG-F^2} \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} d\bar{u}\, d\bar{v}$$

$$= \iint_{X^{-1}(R)} \sqrt{\bar{E}\bar{G}-\bar{F}^2}\, d\bar{u}\, d\bar{v}$$

The integral of f over the region R.

$\int_R f\, dr = \iint_{X^{-1}(R)} f(u,v)\sqrt{EG-F^2}\, du\, dv$ depends the orientation of X.

<u>Theorem</u>. (local baby variation of Gauss-Bonnet Thm.)
Let $X: U \to S$ be an orthogonal parametrization of an oriented surface S, where $U \subseteq \mathbb{R}^2$ homeomorphic to a disk and X is compatible with the orientation of S (i.e $\{X_u, X_v\}$ is positive oriented). Let $R \subseteq X(U)$ be a simple region and $\alpha: I \to S$ $\ni$ $\alpha(I) = \partial R$ (boundary) and is positively oriented, parametrized by arc-length.

2018.05.24 (2)

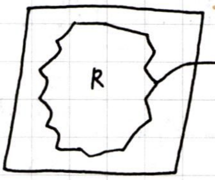
and let $\alpha(s_0), \alpha(s_1), \dots, \alpha(s_k)$ and $\theta_0, \theta_1, \dots, \theta_k$ the vertices and external angles of α

Then $\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g \, ds + \int_R \underline{K} \, d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi$.

$K = \frac{-1}{2\sqrt{EG}} \left(\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right)$

wher k_g is geodesic curvature of the regular arc and

$K \equiv$ Gauss Curvature of S.



On geodesic arc ($k_g = 1$)

$$\int_R K \, ds + \sum_{i=0}^{k} \theta_i = 2\pi$$

pf: Let $u(s), v(s)$ s.t $\alpha(s) = (u(s), v(s))$

$$\therefore k_g = \frac{1}{2\sqrt{EG}} \left(\boxed{-E_v u' + G_u v'} \right) + \frac{d\varphi_i}{ds} \quad (\because F = 0)$$

φ_i angle (α', X_u) on $\alpha[s_i, s_{i+1}]$

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) \, ds = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left(\frac{G_u v'}{2\sqrt{EG}} - \frac{E_v u'}{2\sqrt{EG}} \right) ds + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left(\frac{d\varphi_i}{ds} \right) ds.$$

Recall : Green's formula.

$$\oint_C \left(Q \frac{dv}{ds} + P \frac{du}{ds} \right) = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) du \, dv$$

$A \equiv$ enclosed by curve C.

$$Q = \frac{G_u}{2\sqrt{EG}}, \quad P = \frac{-E_v}{2\sqrt{EG}}$$

By Green's formula, $= \iint_{X^{-1}(R)} \left(\left(\frac{G_u}{2\sqrt{EG}} \right)_u - \left(\frac{-E_v}{2\sqrt{EG}} \right)_v \right) du \, dv$

$$= \iint_{X^{-1}(R)} \left(\left(\frac{G_u}{2\sqrt{EG}} \right)_u + \left(\frac{E_v}{2\sqrt{EG}} \right)_v \right) du \, dv$$

$$= \iint_{X^{-1}(R)} -K\sqrt{EG} \, du \, dv = \int_R -K \, ds$$

$$\because K = \frac{-1}{2\sqrt{EG}} \left(\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right)$$

By the Theorem of turning tangents. 2018.05.24 (3)

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi}{ds}\, ds = \sum_{k=0}^{k} \left[\varphi_i(s_{i+1}) - \varphi_i(s_i) \right]$$

$$= \pm 2\pi - \sum_{i=0}^{k} \theta_i$$

$$= 2\pi - \sum_{i=0}^{k} \theta_i \quad (\because \alpha \text{ is positively oriented.})$$

higher-dimension (S.S chern)

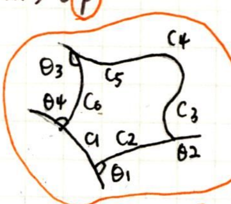
- <u>Global Gauss Bonnet Theorem</u>

R = regular region of an oriented surface S.

($\partial R \cap S$ = closed piecewise curve.)

Let $C_1, C_2, \dots, C_n$ be closed, simple, piecewise regular curves which form ∂R.

Suppose C_i is positively oriented and let $\theta_1, \theta_2, \dots, \theta_p$ be the set of external angle of C_i

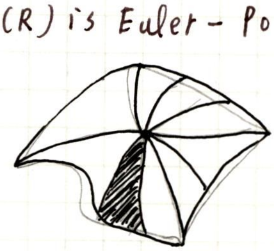


curve 不一定 有 vertex

(有 vertex 才有外角)

Then $\sum_{i=1}^{n} \int_{C_i} k_g(s)\, ds + \int_R K\, d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi \chi(R)$

where $\chi(R)$ is Euler-Poincare characteristic number.

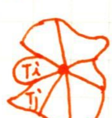


R: simple, closed. $\chi(R) = 1$

— 5/30

<u>Def</u>: A triangulation of a regular region $R \subset S$ is a finite family $\mathcal{T}$ of triangles T_i, $i = 1, \dots, n$ such that

① $\bigcup_{i=1}^{n} T_i = R$



② If $T_i \cap T_j \neq \phi$, then $T_i \cap T_j$ is either common edge T_i and T_j or a common vertex of T_i and T_j.

<u>Def</u>: Euler characteristic number of S relative to the triangulation $\mathcal{T}$ is $\chi(S, \mathcal{T}) = V - E + F = \chi(S)$

V = # of vertices of triangulations

E = # : edge :

F = # : face :

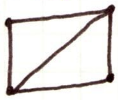
3 vertices

3 edges

1 face

$\chi(R \subseteq \mathbb{R}^2) = V - E + F = 1$

2018.05.30 (2)... $\mathbb{R}^2$



$V=4$
$E=5$
$F=2$

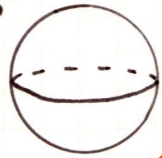
$\chi(R \subseteq \mathbb{R}^2) = 4-5+2 = 1$

<u>Def</u> : A triangulation T_n is invariant under diffeomorphism.

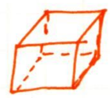
$$\chi(T_n) = V - E + F$$

<u>prop</u> : Every regular region of a regular surface admits a triangulation.

S^2



$V=4$
$E=6$
$F=4$



$V=8$
$E=12$
$F=6$.

$\chi(S^2) = V - E + F = 4-6+4 = 2$ $\qquad$ $\chi(S^2) = 2$

This theorem, if we mistake not, ought to counted among the most elegant in the theory of curved surface (by Gauss)

Pf (Gauss-Bonnet Theorem ; Beauty is the best)

Take a triangulation $\mathcal{T} = \{T_i\}$ of R, $\ni$ each of the T_i must be contained in coordinate Nbhd.

Applying local Gauss-Bonnet Theorem on each T_i and sum over $i = 1, 2, \dots, q = \#\text{faces} = F$

$$\sum_{i=1}^{n} \int_{C_i} k_g\, ds + \int_R K\, d\sigma + \sum_{\substack{1 \le j \le F \\ 1 \le k \le 3}} \theta_{jk} = 2\pi q = 2\pi F \quad - (0)$$

On each T_i,

$$\int_{\partial T_i} k_g\, ds + \int_{T_i} K + \sum_{i=1}^{3} \theta_i = 2\pi$$

$\sum \theta_{jk}$
$\#T_i \leftarrow 1 \le j \le p$
$1 \le k \le 3$

Introduce the interior angles by $\varphi_{jk} = \pi - \theta_{jk}$

$$\sum_{\substack{1 \le j \le F \\ 1 \le k \le 3}} \theta_{jk} = \sum_{\substack{1 \le j \le F \\ 1 \le k \le 3}} (\pi - \varphi_{jk}) = 3\pi F - \sum_{\substack{1 \le j \le F \\ 1 \le k \le 3}} \varphi_{jk}$$

$E_e = \#$ external edges.

$E_i = \#$ internal edges

$V_e = \#$ external vertices

$V_i = \#$ internal vertices.

Fact: $V_e = E_e$, $3F = 2E_i + E_e$

$$\sum_{\substack{1\le j\le F\\1\le k\le 3}} \theta_{jk} = \sum_{\substack{1\le j\le F\\1\le k\le 3}} (\pi - \varphi_{jk}) = 3\pi F - \sum_{\substack{1\le j\le F\\1\le k\le 3}} \varphi_{jk} \xlongequal{\text{by②}} \pi(2E_i + E_e) - \sum_{\substack{1\le j\le F\\1\le k\le 3}} \varphi_{jk}$$

$$= 2\pi E_i + \pi E_e + \underline{\pi E_e - \pi E_e} - \sum_{\substack{1\le j\le F\\1\le k\le 3}} \varphi_{jk} \quad \text{— ◎}$$

$$= 2\pi(E_i + E_e) - \pi V_e - \sum_{\substack{1\le j\le F\\1\le k\le 3}} \varphi_{jk} = 2\pi E - 2\pi V_e - \sum_{\substack{1\le j\le F\\1\le k\le 3}} \varphi_{jk}$$

<u>At V_{0e}</u>: (Interior angles)

在某一個頂點 $= \pi - \theta_{0jk}$

Total interior angle $V_e = \pi V_e - \sum \theta_{jk}$

Interior angles at V (including V_i, V_e)

$$\sum_{\substack{1\le j\le F\\1\le k\le 3}} \varphi_{jk} = 2\pi V_i + \pi V_e - \sum_{i=1}^{p} \theta_i \quad \text{plug into ◎}$$

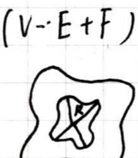
$$\sum_{\substack{1\le j\le F\\1\le k\le 3}} \theta_{jk} = 2\pi E - \pi V_e - 2\pi V_i - 2\pi V_e + \sum_{i=1}^{p} \theta_i$$

$$= 2\pi E - 2\pi V + \sum_{i=1}^{p} \theta_i$$

— 5/31

Gauss-Bonnet Theorem

$$\int_R K\,d\sigma + \int_{\partial C_i} k_g\,ds + \underbrace{\sum_{i=1}^{p} \theta_i}_{\text{external angles}} = 2\pi\chi(R) = 2\pi(V - E + F) \quad \text{— (☆☆)}$$

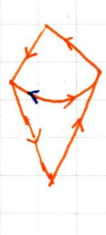


S

<u>Theorem</u>: Gauss-Bonnet

S is <u>compact</u> surface, $\int_S K\,dA = 2\pi\chi(S)$

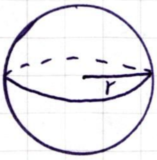
closed



$$\int_r k_g = \int_{-r} k_g. \quad \text{— ☆}$$

2018.05.31(1)

Example: $S^2(r^2)$ compact

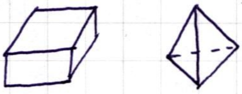


$\underbrace{k_1 = k_2 = \frac{1}{r}}$, $K = \frac{1}{r^2}$

principal curvature

$$\int_{S^2(r)} K\, d\sigma = \frac{1}{r^2} \underbrace{\boxed{\int_{S^2(r)} d\sigma}}_{4\pi r^2} = 4\pi = 2\pi\chi(S^2)$$

$\because \chi(S^2(r)) = 2 = V - E + F$



② Convex Surface S in $\mathbb{R}^3$

$$\int_S K\, d\sigma =$$

Gauss Map $N : S \to S^2$

$dN_p : T_p(S) \to T_{N(p)}(S^2) = T_p(S)$

$K(p) = \det(dN_p) = k_1 k_2$ is the local curvature.

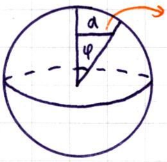
exposion factor for areks under the Gauss map.

By change of variables for multiple integrals.

$$\int_S K\, d\sigma_S = \int_S \det(dN_p)\, d\sigma_S$$

$$= \int_{S^2} d\sigma_{S^2} = \text{Area}(S^2) = 4\pi$$

$$\int_S K\, d\sigma_S = 4\pi \underset{\substack{\uparrow \\ \text{By } ⊛}}{=} 2\pi\chi(S)$$



polar Cap.

On S radius $a = \sin\varphi$

$k = \frac{1}{\sin\varphi}$ (principal) curvature

$k_g^2 + k_n^2 = k^2$

$k_g = \frac{\cos\varphi}{\sin\varphi}$ (recall last time)

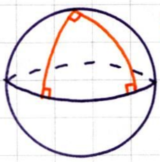
2018.05.31(2)

$$\text{Area of } S = \int_0^{\varphi} 2\pi \sin\theta \, d\theta = 2\pi(1-\cos\varphi)$$

$$\int_S K \, d\sigma = 1 \times (2\pi(1-\cos\varphi)) = 2\pi(1-\cos\varphi)$$

$\because$ geodesic curvature k_g of ∂S is $\dfrac{\cos\varphi}{\sin\varphi}$

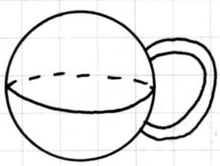
$$\int_{\partial S} k_g \, ds = \int_{\partial S} \frac{\cos\varphi}{\sin\varphi}(2\pi \sin\varphi) = 2\pi\cos\varphi$$



HW
$$\int_S K \, d\sigma + \int_{\partial S} k_g \, ds = 2\pi(1-\cos\varphi) + 2\pi\cos\varphi = 2\pi = 2\pi \underbrace{\chi(S)}_{=1}$$

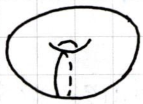
$$\int_K + \cdots + \cdots = 2\pi$$

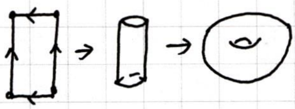
Euler Characteristic number



genus

$\chi(S^2) = 2$





$\chi(T^2) = V - E + F$
$= 1 - 2 + 1 = 0$

$E = 2$
$F = 1$
$V = 1$

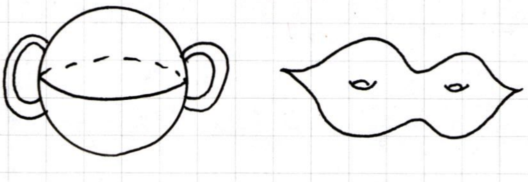
Note : A surface withe Some number of genus

$\chi(S) = 2(1-g)$ $g = \#$ genus

Torus, $g=1$, $\chi(S) = \chi(T^2) = 0$.

Sphere, $g=0$, $\chi(S) = \chi(S^2) = 2$

③ Any compact surface is homeomorphic to a sphere with some number of genus.



Theorem : Diffeomorphic surfaces have same Euler characteristic.
(Homeomorphic)

i.e If S_1 and S_2 are compact, oriented surface, $\chi(S_1) = \chi(S_2)$
Then S_1 is diffeomorphic to S_2
(homeomorphic)

Since the triangulation of a surface1 is mapped to the triangulation of surface S_2 with the same # of vertices, edges and faces.
Euler characteristic # is preserved by homeomorphism.

- S compact, connected, $K = \text{const}$. $\Rightarrow$ S is a sphere
- S compact, $\exists p$ s.t $K(p) > 0$
  $k_1(p) = \max k$
  $k_2(p) = \min k$ } umbillical point

- S compact, $K > 0$, $H = \text{const}$. $\Rightarrow$ S is a sphere

Theorem: Suppose S is compact, oriented surface whose $K(x)>0$ $\forall x \in S$. Then S is diffeomorphic to a standard sphere.

(pf) $\because$ An regular surface must be diffeomorphic to a sphere with same number genus.

By Gauss-Bonnet Theorem (cpt)

$$\underbrace{\int_S K\,d\sigma}_{>0} = 2\pi \chi(S) = 2\pi(2(1-g)) = 4\pi(1-g)$$

$4\pi(1-g)>0$

$g<1$

$\because$ g is a positive integer $\therefore g=0$ $\therefore \chi(S)=2$

$\therefore$ By Theorem, $\chi(S)=\chi(S^2)=2$

$$\underset{\text{diffeophism}}{S \sim S^2}$$

- If a compact surface S is diffeomorphic to the torus, then $\int_S K\,d\sigma = 0$.

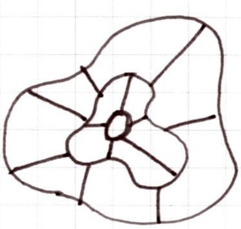
  $\int_S K\,d\sigma = 2\pi\chi(S)$

Four color Map Theorem

1852 F Guthrie
Morgan

1879 Kempe

1970 Appel & Haken 1200.



2018.06.06 (1)

For a given compact surface S, what is the smallest positive integer n s.t every map on S can be n-colored?

Def: The smallest integer n is called the chromatic number of S

The chromatic number of a sphere is 4.

prop. S regular, cpt, orientable surface is not homeomorphic to a sphere, prove there are points on S where the K (Gauss Curvature) is positive, negative and zero. — 6/6

$$\int_S K d\sigma = 2\pi \chi(S) = 2\pi(2-2g)$$

S is homeomorphic to a sphere adding # genus.

∵ S is not homeomorphic to a sphere.

$2(1-g) \neq 2$, $g \neq 0$.

g is positive (including zero) integer.

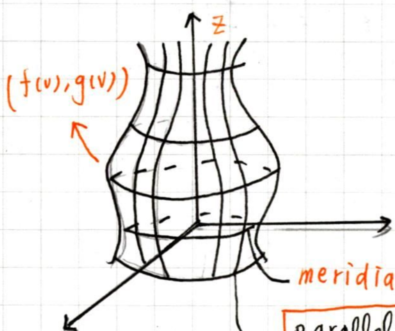
∴ $\int_S K d\sigma < 4\pi(1-g) \leq 0$.

∵ S cpt, $\exists$ a point $p \in S$, $K(p) > 0$.

∴ K must attain a negative at some point

※ K is cont. ∴ By intermediate value Theorem, we conclude $K=0$ at some point.

• Clairant's Theorem



meridians are geodesic

$\boxed{\text{parallels are geodesic if } f'=0}$

$$X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$$

$f > 0$

$f'^2 + g'^2 = 1$

- Clairaut's Theorem

Let α be a curve parametrized by arc-length on a surface of revolution S. let f be the distance function of a point of S from the axis of rotation, and φ be the angle between α' and meridian of S.

If α is geodesic, then $f\cos\varphi = \text{const.}$

Conversely, if $f\cos\varphi = \text{const.}$ along α & if no part of α is part of parallels of S, then α is geodesic.

"pf": $X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$ $f(v) > 0$, $f'^2 + g'^2 = 1$

X_u, X_v

$E = f^2(v),\ F = 0,\ G = 1$ (recall)

The geodesic equations.

$$\frac{d}{ds}(Eu' + Fv') = \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2)$$

$s \rightarrow \because$ parametrized by arc-length

$$\frac{d}{ds}(Fu' + Gv') = \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2)$$

$$\begin{cases} \dfrac{d}{ds}(f^2 u') = 0 & \text{— ①} \\ \dfrac{d}{ds}(v') = ff' u'^2 & \text{— ②} \end{cases}$$

- 1^{st} f.f $\rightarrow$ $f^2 u'^2 + v'^2 = 1$. $(|\alpha'|^2 = Eu'^2 + 2Fu'v' + Gv'^2)$. $f = f(v)$

Let $\{X_u f^{-1}, X_v\}$ be a basis s.t $\alpha' = \cos\varphi\, f^{-1} X_u + \sin\varphi\, X_v$

$= X_u u' + X_v v'$

$\underline{u' = f^{-1}\cos\varphi}$, $v' = \sin\varphi$

$\downarrow$

$u' f^2 = f\cos\varphi$

$\Downarrow$

$\underline{u' f^2(v)} = f\cos\varphi$

$\downarrow$ by ① $\dfrac{d}{ds}(u' f^2(v)) = 0$

$f\cos\varphi = \text{const}$.

Conversely, $\rho\cos\psi = \text{const.} = \tilde{c}$ for some constant $\tilde{c}$ along curve α

$\rho\cos\psi = \rho^2 u' = \rho^2 f^2(v)$

We have $\frac{d}{ds}(\rho^2 f^2(v)) = 0$.

Now, we need to prove $\frac{d}{ds}(v') = ff'u'^2$ holds.

$u' = \frac{\cos\psi}{\rho} = \frac{\tilde{c}}{\rho^2}$, $(u')^2 = \frac{(\tilde{c})^2}{\rho^4}$

$\because f^2(v)u'^2 + v'^2 = 1$, $v'^2 = 1 - f^2(v)u'^2 = 1 - \underbrace{f^2(v)u'}_{\tilde{c}}u' = 1 - \frac{\tilde{c}^2}{\rho^2}$

$$2v'v'' = \frac{-\rho^2 \cdot (\cancel{\tilde{c}^2})' + \tilde{c}^2(2\rho \frac{\partial \rho}{\partial v}\frac{\partial v}{\partial s})}{\rho^4} = \frac{2\tilde{c}^2 \frac{d\rho}{dv} v'}{\rho^3}$$

$$(v'' - \frac{\tilde{c}^2 \frac{d\rho}{dv}}{\rho^3})v' = 0.$$

$\because \frac{\tilde{c}^2}{\rho^3} = \rho(u')^2$, $(v'' - \rho\frac{d\rho}{dv}(u')^2)v' = 0$

$(v'' - ff'(u')^2)v' = 0.$, $\rho = f(v)$

$v' \neq 0 \Rightarrow v'' - ff'(u')^2 = 0.$

■ Presudo - Sphere.

Surface of revolution. $X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$

$f>0$, $f'^2 + g'^2 = 1$

$E = f^2$, $F = 0$, $G = 1$.

$$K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) = \frac{-f''}{f} \text{ or } \left(= \frac{eg - f^2}{EG - F^2}\right)$$

$$\begin{cases} K = \frac{-f''}{f} & \text{— ①} \\ f'^2 + g'^2 = 1 & \text{— ②} \end{cases}$$

case I. $K = 0$. $f'' = 0$, f is linear
$f = av + b$.

by ②, $g' = \pm\sqrt{1-a^2}$

If $a = 0$, this is a circular cylinder
$X(u,v) = (\cos u, \sin u, v)$

$|a| = 1$, X-Y plane

$0 < |a| < 1$, it is circular cone

Case II $K>0$, $K=\frac{1}{R^2}$

$$\frac{1}{R^2}=\frac{-f''}{f} \Rightarrow -f''=R^2 f$$

$$f'^2+g'^2=1 \longleftarrow f(v)=a\cos\left(\frac{v}{R}+b\right)$$

$$g(v)=\int\sqrt{1-\frac{a^2}{R^2}\sin^2\left(\frac{v}{R}\right)}\,dv$$

$a=0$, it does not give a surface

$a=R$, $f(v)=R\cos\left(\frac{v}{R}\right)$

$$g(v)=R\sin\left(\frac{v}{R}\right)$$

$\therefore$ We have a sphere with radius R

Case III $K<0$, $K=-1$

$$\begin{cases} f''-f=0 \\ f'^2+g'^2=1 \end{cases} \qquad f=ae^{v}+be^{-v}$$

$a=1$, $b=0$:

$$g(v)=\int\sqrt{1-e^{2v}}\,dv \qquad \text{Let } \cos\theta=e^{v}$$

$$-\sin\theta\,d\theta=e^{v}dv=\cos\theta\,dv$$

$$=-\int\frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$=\sin\theta-\ln(\sec\theta+\tan\theta)$$

$$=\sqrt{1-e^{2v}}-\ln\left(e^{-v}+\sqrt{e^{-2v}-1}\right)$$

$$X(u,v)=\left(e^{v}\cos u, e^{v}\sin u, \sqrt{1-e^{2v}}-\underline{\ln\left(e^{-v}+\sqrt{e^{-2v}-1}\right)}\right)$$

$\parallel$ $\cosh^{-1}(e^{-v})$

$$=\left(e^{v}\cos u, e^{v}\sin u, \sqrt{1-e^{2v}}-\cosh^{-1}(e^{-v})\right)$$

$\Rightarrow K=-1$ (double check) $\boxed{\text{presudo-sphere}}$

$E=e^{2v}$, $F=0$, $G=1$ metric $e^{v}du^2+dv^2$

Let $w=e^{-v}$, $\tilde{X}(u,w)=\left(\frac{1}{w}\cos u, \frac{1}{w}\sin u, \sqrt{1-w^2}-\cosh^{-1}(w)\right)$

$$\tilde{X}_u=\left(-\frac{1}{w}\sin u, \frac{1}{w}\cos u, 0\right)$$

$$\tilde{X}_w=$$

$$\tilde{E}=\frac{1}{w^2},\ \tilde{F}=0,\ \tilde{G}=\frac{1}{w^2} \qquad \frac{du^2+dw^2}{w^2}$$

$U = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

$I = \dfrac{dx^2 + dy^2}{y^2}$

※ S regular, compact, connected, oriented surface

S homeomorphic to a sphere, $\exists$ some points s.t. $K>0$, $K<0$ and $K=0$

$K=-1$, $(f(v)\cos u, f(v)\sin u, g(v))$

$X(u,v) = (e^v \cos u, e^v \sin u, \sqrt{1-e^{2v}} - \cosh^{-1}(e^{-v}))$. pesudo-sphere

Let $w = e^{-v} (= \frac{1}{f})$

$X(u,w) = (\frac{1}{w}\cos u, \frac{1}{w}\sin u, \sqrt{1-\frac{1}{w^2}} - \cosh^{-1}(w))$

$E = \frac{1}{w^2}$

$F = 0$ metric form $\dfrac{du^2 + dw^2}{w^2} = Eu'^2 + 2Fu'v' + Gv'^2$

$G = \frac{1}{w^2}$

isometry, $\exists \varphi: P_S \to U$

$\varphi(I_p(\alpha')) = I_{\varphi(p)}(\varphi(\alpha'))$

$U \equiv$ upper half plane

$\|$

$\{(x,y) \in \mathbb{R}^2 \mid y>0\}$ its 1st fundamental form $\dfrac{dx^2+dy^2}{y^2}$

$K_U = -1$

- Hyperbolic models

Poincare disc $D = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$

its 1st fundmental form $= \dfrac{4}{(1-(u^2+v^2))^2}(du^2+dv^2) = I_D = \dfrac{4}{(1-|z|^2)^2} dz\, d\bar{z}$

$z = u + iv$

$E = \dfrac{4}{(1-(u^2+v^2))^2} = G$, $F=0$.

$K = \dfrac{-1}{2\sqrt{EG}}\left(\left(\dfrac{E_v}{\sqrt{EG}}\right)_v + \left(\dfrac{G_u}{\sqrt{EG}}\right)_u\right) = -1$

Upper half-plane U, $\dfrac{dx^2+dy^2}{y^2}$, $E = \frac{1}{y^2} = G$, $F=0$.

$w = x + iy$

check

2018.06.07 (2)

$\varphi: D \to U$ is isometry.

$z \to w = i\,\frac{1-z}{1+z}$ $\quad \varphi(z) = i\frac{(1-z)}{1+z}$

$\therefore I_U = \frac{dx^2+dy^2}{y^2} = \frac{dw\,d\bar{w}}{|\mathrm{Im}\,w|^2}$ $\quad \bar{w} = x - iy$

※ PS : pesudo-sphere

PS $\quad X(u,w) = \left(\frac{1}{w}\cos u, \frac{1}{w}\sin u, \underline{\sqrt{1-\frac{1}{w^2}}} - \cosh^{-1}(w)\right)$

$w>1$ (well-defined)

$I_{ps} = \frac{du^2+dw^2}{w^2}$ $\quad , w = e^{-v},\ f(u) = e^{v}$

Note $w>1$ s.t $X(u,v)$ well-defined & smooth

$\alpha(s) = X(u(s), w(s))$

$1 = |\alpha'(s)|^2 = \frac{du^2+dw^2}{w^2}$

$\begin{cases} u'^2 + w'^2 = w^2 \text{ — ①} \\ f\cos\varphi = \text{constant} = \tilde{c} \end{cases}$ $\quad , f = f(v) = \frac{1}{w} \Rightarrow \left(\frac{1}{w}\right)\left(\frac{u'}{w}\right) = \tilde{c}$

$\Rightarrow u' = \tilde{c}w$ — ②

by ②, $u' = \tilde{c}w^2$

case I : $\tilde{c} = 0$, $u' = $ constant.

plug into ①. $w'^2 = w^2$, $w = e^s$

case II : $\tilde{c} \neq 0$.

In ① $(\tilde{c}w^2)^2 + (w')^2 = w^2$

$w' = \pm w\sqrt{1-\tilde{c}^2w^2}$ — ③

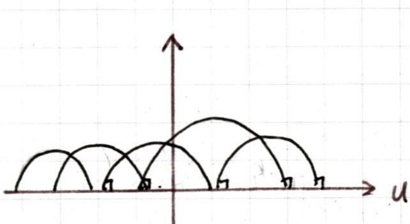
$\frac{u'}{w'} = \frac{②}{③} = \frac{\tilde{c}w}{\sqrt{1-\tilde{c}^2w^2}}$, $\quad u' = \frac{\tilde{c}w}{\sqrt{1-\tilde{c}^2w^2}}w'$

$u - C_0 = \pm\int \frac{\tilde{c}w}{\sqrt{1-\tilde{c}^2w^2}}dw = \mp\frac{1}{\tilde{c}}\sqrt{1-\tilde{c}^2w^2}$

$\underline{(u-C_0)^2 + w^2 = \frac{1}{\tilde{c}^2}}$

C_0 = constant

circle, center at C_0.

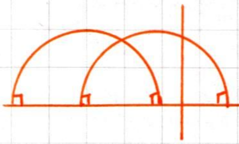


semi-circle

2018.06.07 (3)

<u>Note</u>: The geodesics are the image under X of the parts of the circles in u-w planes and lying in the region $w > 1$

These circles all have centre on the u-axis and intersect the u-axis perpendicular



$U = \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$

$I_u \equiv$ its 1^{st} f.f. $= \dfrac{du^2 + dv^2}{v^2}$

$$E = \frac{1}{v^2},\ F = 0,\ G = \frac{1}{v^2}$$

$|\alpha'| = 1,\ u'^2 + v'^2 = v^2$

By geodesic equations,

$$\frac{d}{ds}(Eu' + Fv') = \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2)$$

$$\frac{d}{ds}(Fu' + Gv') = \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2)$$

$$\frac{d}{ds}\left(\frac{u'}{v^2}\right) = 0 \quad \text{—— ①}$$

$$\frac{d}{ds}\left(\frac{v'}{v^2}\right) = \frac{1}{2}\left(\left(\frac{1}{v^2}\right)' u'^2 + \left(\frac{1}{v^2}\right)' v'^2\right) = -\frac{1}{v^3}(u'^2 + v'^2)$$

$$= \frac{-1}{v^3} v^2 = \frac{-1}{v}$$

$$\frac{d}{ds}\left(\frac{v'}{v^2}\right) = \frac{-1}{v} \quad \text{— ②} \Rightarrow LHS = \frac{v^2 v'' - v'(2vv')}{v^4} = \frac{vv'' - 2vv'^2}{v^3} = \frac{-1}{v} = RHS$$

by ①

$$\frac{v^2(u'') - u'(2vv')}{v^4} = 0.$$

$$\begin{cases} vu'' - 2u'v' = 0. & \text{— ④} \\ vv'' - 2v'^2 + v^2 = 0 & \text{— ⑤} \end{cases}$$

The geodesics in the upper-half plane is of two forms.

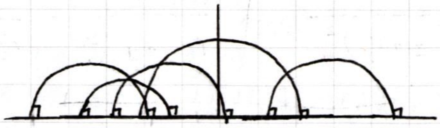
Ⅰ ∵ $\dfrac{d}{ds}\left(\dfrac{u'}{v^2}\right) = 0 \Rightarrow u' = cv^2$. If $c = 0 \Rightarrow u = \text{const}$.

Ⅱ By ④.⑤.
$$u(s) = c\tanh(cs) + c_0$$
$$v(s) = c\,\mathrm{sech}(cs)$$

$(u-c_0) = c\tanh(s)$

$v = c\,\mathrm{sech}(s)$

$(u-c_0)^2 + v^2 = c^2 \underbrace{(\tanh^2(s) + \mathrm{sech}^2(s))}_{=1}$



– 6/13

Def : Mobius transformation on the upper half plane are functions of the form $f(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, $cz+d \neq 0$, $z = -\dfrac{d}{c}$

$U = \{ (x,y) \in \mathbb{R}^2 \mid y > 0 \}$, upper half plane.

$f : U \to U$ isometry ?

$I_z(\alpha') \overset{?}{=} I_{f(z)}(f(\alpha'))$ — (*)

$f(z) = \dfrac{az+b}{cz+d}$, $z = x+iy$

$f(\underbrace{x+iy}_{\in U}) = \dfrac{a(x+iy)+b}{c(x+iy)+d} = \dfrac{(ax+b)+iay}{(cx+d)+icy} \dfrac{(cx+d)-icy}{(cx+d)-icy}$

$= \dfrac{acx^2 + (ad+bc)x + acy^2 + bd}{(cx+d)^2 + c^2y^2} + i \boxed{\dfrac{y}{(cx+d)^2 + c^2y^2}}$

$\mathrm{Im}(f(x+iy)) > 0$.

1st f.f for U $\dfrac{dx^2+dy^2}{y^2}$

$\because$ $z = x+iy$, $\bar{z} = x-iy$, $dz = dx+idy$
$d\bar{z} = dx-idy$

$dz\,d\bar{z} = (dx+idy)(dx-idy) = dx^2+dy^2$

$z - \bar{z} = 2iy$, $y = \dfrac{z-\bar{z}}{2i}$, $y^2 = \dfrac{(z-\bar{z})^2}{-4}$

$ds^2 = \dfrac{-4\,dz\,d\bar{z}}{(z-\bar{z}^2)}$ 1st f.f on Domain f.

" 1st f.f on range of f.

Now the 1^{st} f.f on the range of f is given by

$$ds^2 = \frac{-4\, d(f(z))\, d(f(\bar{z}))}{(f(z)-f(\bar{z}))^2} = \frac{-4 f'(z) f'(\bar{z})\, dz\, d\bar{z}}{(f(z)-f(\bar{z}))} \quad - ⊛$$

$$f(z) = \frac{az+b}{cz+d}, \quad f'(z) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

$$f(\bar{z}) = \frac{a\bar{z}+b}{c\bar{z}+d}, \quad f'(z) = \frac{(c\bar{z}+d)a-(a\bar{z}+b)c}{(c\bar{z}+d)^2} = \frac{1}{(c\bar{z}+d)^2} \quad — ①$$

$$f(z)-f(\bar{z}) = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d} = \frac{z-\bar{z}}{(cz+d)(c\bar{z}+d)} \quad — ②$$

plug ①, ② into ⊛

$$ds^2 = \frac{-4\left(\frac{1}{(cz+d)^2}\right)\left(\frac{1}{(c\bar{z}+d)^2}\right) dz\, d\bar{z}}{\left(\frac{z-\bar{z}}{(cz+d)(c\bar{z}+d)}\right)^2} = \frac{-4\, dz\, d\bar{z}}{(z-\bar{z})^2} = \text{1st f.f on the domain}$$

$\therefore f: U \to U$ is an isometry.

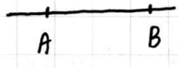
$K(U) = -1$

$f: U_1 \to U_2$ isometry $\Rightarrow K(U_1) = K(U_2)$

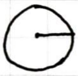
- 5 postulates:

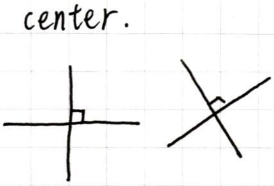
Euclidean:

① A straight line can be drawn joining any two points

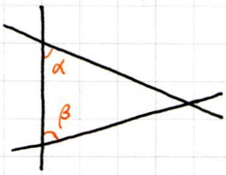


② Any straight line can be extended indefinitely in a straight line.

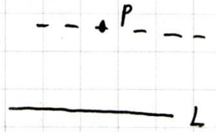
③  Given any straight segment, a circle can be drawn having the segment as radius and one of end points as center.

④  all right angles are congruent.

⑤



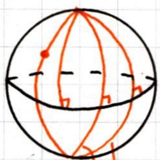
$\alpha+\beta < 90^\circ + 90^\circ$



In a plane, given a line and a point not on it,
at most one parallel ~~parall~~ to L (passing through P).

<u>1817</u> Gauss, Riemann
Lobachesky, Boulier, Beltrani

<u>Sphererical Space</u>



great circle (geodesic)

"∩" 2pts.

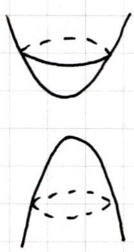
parallel

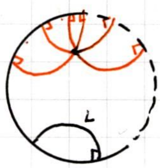
circumference $< 2\pi r$

$\Sigma \varphi_i > \pi$

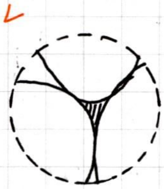
(Poincare Disk model)

<u>hyperbolic space</u> stereographic projection





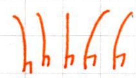
∞ // L



$c > 2\pi r$

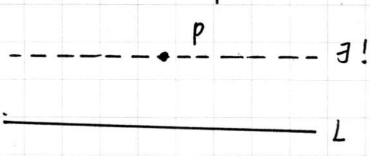
$\Sigma \varphi_i < \pi$

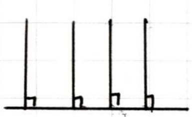
internal angles



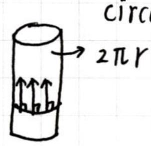
divergent

Euclidean Space





circumference



S Surface

- S cpt, connected, oriented

  $K = \text{const.} \Rightarrow$ sphere

  S cpt, $\forall p \in S$, $K(p) > 0$

  S is homeomorphic to sphere
  diffeomorphic

sectional curvature = Gauss Curvature.